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Paired Many-to-Many 3-Disjoint Path Covers in Bipartite Toroidal Grids

Jung-Heum Park*
School of Computer Science and Information Engineering, The Catholic University of Korea, Bucheon, Korea
j.h.park@catholic.ac.kr

Abstract
Given two disjoint vertex-sets, $S = \{s_1,...,s_k\}$ and $T = \{t_1,...,t_k\}$ in a graph, a paired *many-to-many k-disjoint path cover* joining S and T is a set of pairwise vertex-disjoint paths $\{P_1,...,P_k\}$ that altogether cover every vertex of the graph, in which each path P_i runs from s_i to t_i. In this paper, we first study the disjoint-path-cover properties of a bipartite cylindrical grid. Based on the findings, we prove that every bipartite toroidal grid, excluding the smallest one, has a paired many-to-many 3-disjoint path cover joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if and only if the set $S \cup T$ contains the equal numbers of vertices from different parts of the bipartition.

Category: Algorithms and Complexity

Keywords: Disjoint path; Path cover; Path partition; Cylindrical grid; Torus

I. INTRODUCTION

Let G be a finite, simple undirected graph whose vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. A *path* from $v \in V(G)$ to $w \in V(G)$, referred to as a v–w path, is a sequence $\langle u_1, ..., u_l \rangle$ of distinct vertices of G such that $u_1 = v$, $u_l = w$, and $(u_i, u_{i+1}) \in E(G)$ for all $i \in \{1,...,l-1\}$. If $l \geq 3$ and $(u_l, u_1) \in E(G)$, the sequence is called a *cycle*. A path that visits each vertex exactly once is a *Hamiltonian path*; a cycle that visits each vertex exactly once is a *Hamiltonian cycle*. A *path cover* of a graph G is a set of paths in G such that every vertex of G is contained in at least one path. A *disjoint path cover* (DPC for short) of G is a set of disjoint paths that altogether cover every vertex of G. This paper is concerned with a DPC in which each path runs from a prescribed source to a prescribed sink.

Given disjoint subsets $S = \{s_1,...,s_k\}$ and $T = \{t_1,...,t_k\}$ of $V(G)$ for a positive integer k, a *many-to-many k-disjoint path cover* is a DPC composed of k paths that collectively join S and T; if each source $s_i \in S$ must be joined to a specific sink $t_i \in T$, the DPC is called *paired*, and it is *unpaired* if no such constraint is imposed. Refer to Fig. 1 for examples.

There are two other DPC types: A *one-to-many k-disjoint path cover* for $S = \{s\}$ and $T = \{t_1,...,t_k\}$ is a DPC made of k paths, each of which joins a pair of source s and sink t_i, $i \in \{1,...,k\}$; when $S = \{s\}$ and $T = \{t\}$, a DPC composed of k paths, each of which joins s and t, is named a *one-to-one k-disjoint path cover*. As is intuitively clear, we will call the vertices in S and in T *sources* and *sinks*, respectively, which together form a set of *terminals.*

The existence of a disjoint path cover in a graph is closely related to the Hamiltonian properties, as well as the concept of vertex connectivity, which was characterized in terms of the minimum number of disjoint paths. For

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*Corresponding Author

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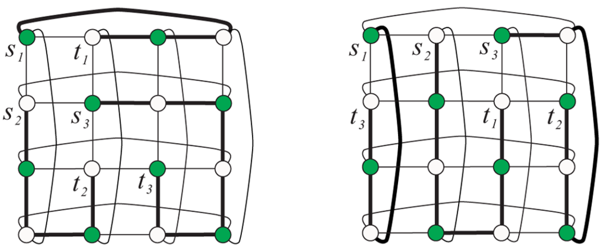


(a) Paired 3-DPC (b) Unpaired 3-DPC

Fig. 1. Examples of many-to-many disjoint path covers.

instance, a Hamiltonian cycle forms a one-to-one 2-DPC joining $\{s\}$ and $\{t\}$ for every pair of distinct vertices s and t. Disjoint path cover problems are applicable in many areas such as software testing, database design, and code optimization [1, 2]. In addition, the problem is concerned with applications where the full utilization of network nodes is important [3]. The problems have been studied for various classes of graphs, such as interval graphs [4, 5], hypercubes [6-8], torus networks [9-12], dense graphs [13], and cubes of connected graphs [14, 15].

In the context of the Hamiltonian path problem, the rectangular grid first appeared in the literature in [16]. In the formal definition of the $m \times n$ rectangular grid, the vertices are often chosen from the points of the Euclidean plane with integer coordinates so that the vertices and edges form a rectangular grid with n vertices appearing in each of m rows and m vertices in each of n columns.

DEFINITION 1 (Rectangular grid). *The* $m \times n$ *rectangular grid* G *is a graph such that* $V(G) = \{v_j^i : 0 \le i \le m-1, 0 \le j \le n-1\}$ *and* $E(G) = \{(v_j^i, v_{j'}^{i'}): |i - i'| + |j - j'| = 1\}$.

Besides the rectangular grid graph, there are two related classes of grid graphs: The $m \times n$ cylindrical grid is constructed from the $m \times n$ rectangular grid by adding horizontal wrap-around edges (v_{n-1}^i, v_0^i) for $i \in \{0, ..., m-1\}$; the toroidal grid can be generated from the $m \times n$ cylindrical grid by adding vertical wrap-around edges (v_j^{m-1}, v_j^0) for $j \in \{0, ..., n-1\}$.

DEFINITION 2 (Cylindrical grid). *The* $m \times n$ cylindrical grid G *is a graph such that* $V(G) = \{v_j^i : 0 \le i \le m-1, 0 \le j \le n-1\}$ and $E(G) = \{(v_j^i, v_{j'}^{i'}): (j = j' \ \& \ |i - i'| = 1)$ or $(i = i' \ \& \ j' \equiv j + 1 \pmod{n})\}$, *where* $n \ge 3$.

DEFINITION 3 (Toroidal grid). *The* $m \times n$ toroidal grid G *is a graph such that* $V(G) = \{v_j^i : 0 \le i \le m-1, 0 \le j \le n-1\}$ and $E(G) = \{(v_j^i, v_{j'}^{i'}): (j = j' \ \& \ i' \equiv i + 1 \pmod{n})$ or $(i = i' \ \& \ j' \equiv j + 1 \pmod{n})\}$, *where* $m, n \ge 3$.

The rectangular grid is a bipartite graph and thus its vertices may be colored in two colors, green and white, in such a way that every pair of adjacent vertices is colored differently (hereafter, we will denote the color of vertex v by $c(v)$). In contrast, the $m \times n$ cylindrical grid is bipartite if and only if n is even; the $m \times n$ toroidal grid is bipartite if and only if both m and n are even. Each of the bipartite cylindrical and toroidal grids is *balanced* in a way that its two color classes have equal cardinality. We will also call a subset of $V(G)$ *balanced* if the number of vertices in the subset that belong to each of the two color classes is equal.

The existence of a paired (many-to-many) 2-DPC in a bipartite toroidal grid was studied, as shown below:

THEOREM 1 (Makino [17]). *An* $m \times n$ *toroidal grid with* $m, n \ge 4$, *both even, has a paired* 2*-DPC for a pair of terminal sets S and T if and only if their union is balanced.*

THEOREM 2 (Park and Ihm [18]). *For an* $m \times n$ *toroidal grid G with* $m, n \ge 4$, *both even, and an arbitrary edge* e_f *of G, the subgraph,* $G - e_f$, *of G with* e_f *being deleted has a paired* 2*-DPC joining S and T if and only if* $S \cup T$ *is balanced*.

THEOREM 3 (Kim and Park [19]). *For an* $m \times n$ *toroidal grid G with* $m, n \ge 4$, *both even, and an arbitrary vertex* v_f *of G, the subgraph,* $G - v_f$, *of G with* v_f *being deleted has a paired* 2*-DPC joining S and T if and only if one of the four terminals in* $S \cup T$ *has the same color as* v_f *and the other three have a different color from* v_f.

In this paper, we prove that an $m \times n$ bipartite toroidal grid with $(m, n) \ne (4, 4)$ has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if and only if $S \cup T$ is balanced. The proof is based on certain disjoint-path-cover properties of a bipartite cylindrical grid (investigated in Section III), as well as the necessary and sufficient condition for a bipartite cylindrical grid to have a paired 2-DPC joining S and T (established in [18]).

II. NOTATION AND PREVIOUS WORKS

For an $m \times n$ grid graph, whether rectangular, cylindrical, or toroidal, R_i denotes the vertex set $\{v_j^i : 0 \le j \le n-1\}$ of row i, whereas C_j denotes the vertex set $\{v_j^i : 0 \le i \le m-1\}$ of column j, implying that v_j^i is the vertex in both row i and column j. Based on these notations, we respectively indicate multiple rows and columns as $R_{i,i'} = \cup_{i \le r \le i'} R_r$ if $i \le i'$; $R_{i,i'} = \emptyset$ otherwise, and $C_{j,j'} = \cup_{j \le r \le j'} C_r$ if $j \le j'$; $C_{j,j'} = \emptyset$ otherwise. All arithmetic on the indices of vertices of the cylindrical and toroidal grids is done modulo n or m as needed.

The Hamiltonian properties of the rectangular and cylindrical grids have been revealed in previous studies, some of which will be effectively used to derive our results. A bipartite graph that is balanced is called *Hamiltonian-laceable* if there is a Hamiltonian path between any two vertices from different color classes [20]. The concept of Hamiltonian-laceability has often been extended in such a way that a bipartite graph whose color classes may differ in cardinality by exactly one is also *Hamiltonian-laceable* if every pair of vertices from the same major color class can be joined by a Hamiltonian path. Finally, a bipartite graph G is called 1-*fault Hamiltonian-laceable* if G remains Hamiltonian-laceable, even if a single vertex or edge is deleted from G.

LEMMA 1 (Chen and Quimpo [21]). *Let G be an $m \times n$ rectangular grid with $m, n \geq 2$.* (a) *If mn is even, then G has a Hamiltonian path from a corner vertex, i.e., a vertex of degree two, to any other vertex in the different color class.* (b) *If mn is odd, then G has a Hamiltonian path from a corner vertex to any other vertex in the same color class.*

LEMMA 2 (Tsai, Tan, Chuang, and Hsu [22]). *An $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$ is* 1-*fault Hamiltonian-laceable.*

A necessary and sufficient condition was established by Park and Ihm [18] for an $m \times n$ bipartite cylindrical grid to have a paired 2-DPC joining disjoint terminal sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$; furthermore, *inadmissible configurations* of the four terminals which would not permit a paired 2-DPC in the cylindrical grid were classified as one of four cases: (i) $m \geq 4$ & even $n \geq 6$, (ii) $n = 4$, (iii) $m = 2$ & even $n \geq 6$, and (iv) $m = 3$ & even $n \geq 6$, as shown in Lemmas 3 through 6.

LEMMA 3. *For $m \geq 4$ and even $n \geq 6$, an $m \times n$ cylindrical grid G has a paired* 2-*DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B0, or C0:*

- **A0:** $s_1 = v_i^0$, $s_2 = v_p^0$, $t_1 = v_j^0$, *and* $t_2 = v_q^0$ *for some i, j, p, and q such that* $i < p < j < q$;
- **B0:** $s_1 = v_i^r$, $t_1 = v_{i+1}^{r+1}$, $s_2 = v_{i+1}^r$, *and* $t_2 = v_i^{r+1}$ *for some i and r;*
- **C0:** $s_1 = v_i^0$, $t_1 = v_{i+1}^1$, $t_2 = v_{i+2}^1$, *and* $s_2 = v_{i+3}^0$ *for some i.*

LEMMA 4. *For $m \geq 2$, an $m \times 4$ cylindrical grid G has a paired* 2-*DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A1, B0, or C1:*

- **A1:** $s_1, t_1 \in R_{r_1}$, $s_2, t_2 \in R_{r_2}$, *and* $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ *for some* r_1 *and* r_2;
- **C1:** $s_1 = v_i^r$, $t_1 = v_{i+1}^{r+1}$, $t_2 = v_{i+2}^{r+1}$, *and* $s_2 = v_{i+3}^r$ *for some i and r.*

LEMMA 5. *For even $n \geq 6$, a $2 \times n$ cylindrical grid G has a paired* 2-*DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B2, C2, or D2:*

- **B2:** $S \cup T = \{v_i^0, v_i^1, v_j^0, v_j^1\}$ *and* $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ *for some i and j with* $i \neq j$;
- **C2:** $s_1 = v_i^0$, $t_1 = v_j^1$, $s_2 = v_p^0$, $t_2 = v_q^1$, *and* $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ *for some i, j, p, and q such that* $i < p < j < q$.
- **D2:** $s_1 = v_i^0$, $s_2 = v_p^0$, $t_1 = v_j^1$, $t_2 = v_q^1$, *and* $c(s_1) = c(s_2) \neq c(t_1) = c(t_2)$ *for some i, j, p, and q such that* $i < p < j < q$.

LEMMA 6. *For even $n \geq 6$, a $3 \times n$ cylindrical grid G has a paired* 2-*DPC joining $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$ if and only if $S \cup T$ is balanced, and the four terminals in $S \cup T$ do not form an inadmissible configuration equivalent to A0, B0, C3, D3, E3, or F3:*

- **C3:** $s_1 = v_i^0$, $t_1 = v_j^1$, $t_2 = v_q^1$, $s_2 = v_p^0$, *and* $c(s_1) = c(t_1) \neq c(s_2) = c(t_2)$ *for some i, j, p, and q such that* $i < j < q < p$, $q = j + 1$, *and* $(n - 1 - p) + i \geq 2$;
- **D3:** $s_1 = v_i^1$, $s_2 = v_p^1$, $t_1 = v_j^1$, $t_2 = v_q^1$, *and* $c(s_1) = c(t_2) \neq c(t_1) = c(s_2)$ *for some i, j, p, and q such that* $i < p < j < q$, $p = i + 1$, *and* $q = j + 1$;
- **E3:** $s_1 = v_i^0$, $s_2 = v_p^0$, $t_2 = v_q^2$, $t_1 = v_j^2$, *and* $c(s_1) = c(s_2) \neq c(t_1) = c(t_2)$ *for some i, j, p, and q such that* $i < p < q < j$, $q - p - 1 \geq 2$, *and* $(n - 1 - j) + i \geq 2$;
- **F3:** $s_1 = v_i^0$, $t_2 = v_q^2$, $s_2 = v_p^0$, $t_1 = v_j^2$, *and* $c(s_1) = c(t_2) \neq c(s_2) = c(t_1)$ *for some i, j, p, and q such that* $q' < j'$, $j' - q' - 1 \geq 2$, and $(n - 1 - p') + i' \geq 2$, where $i' = \min\{i, q\}$, $q' = \min\{i, q\}$, $j' = \min\{j, p\}$, and $p' = \min\{j, p\}$.

REMARK 1. The four terminals in $S \cup T$ form an inadmissible configuration in a bipartite cylindrical grid only if each row contains an even number of terminals.

III. DISJOINT PATH COVERS IN BIPARTITE CYLINDRICAL GRIDS

Suppose that disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are given in an $m \times n$ bipartite toroidal grid. If we divide the toroidal grid into two cylindrical grids, $m_1 \times n$ and $m_2 \times n$ cylindrical grids for some $m_1, m_2 \geq 2$ with $m_1 + m_2 = m$, then each cylindrical grid may have an "incomplete" terminal set in a sense that s_i is contained in its terminal set but t_i is not for some $i \in \{1, 2, 3\}$, and vice versa. In this section, we derive certain useful properties of a disjoint path cover in a bipartite cylindrical grid with an incomplete terminal set, where the notion of a disjoint path cover is "generalized" in a way that allows for a one-vertex path (Note that a disjoint path cover joining disjoint terminal sets S and T contains no one-vertex path). A *boundary* row in an $m \times n$ cylindrical grid hereafter refers to row 0 or row $m - 1$.

THEOREM 4. *Let G be an m × n cylindrical grid with $m \geq 2$ and even $n \geq 4$, in which three distinct terminals s_1, $s_2 \in S$ and $t_1 \in T$ are given such that not all the three are of the same color. Then, there exist two disjoint paths, s_1–t_1* and *s_2–x paths, possibly $x = s_2$, that altogether cover all the vertices of G*

- *for every vertex x in one boundary row and for at least one vertex x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced, or*
- *for every vertex x except one in one boundary row and for at least two vertices x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced.*

Proof. Suppose we are given three distinct terminals s_1, t_1, and s_2 in G such that the three are not of the same color. Then, there is a terminal with a color different from the other two, so $\{s_1, t_1, s_2, x\}$ is balanced if and only if x has the same color as the terminal. In addition, inspecting the inadmissible configurations in each of the four cases, where (i) $m \geq 4$ & even $n \geq 6$, (ii) $n = 4$, (iii) $m = 2$ & even $n \geq 6$, and (iv) $m = 3$ & even $n \geq 6$, can reveal that there exists an inadmissible configuration Z such that for every vertex $x \in V(G) \setminus \{s_1, t_1, s_2\}$, the four terminals in $\{s_1, t_1, s_2, x\}$ do not form an inadmissible configuration, or form an inadmissible configuration equivalent only to Z, i.e., the four terminals do not form an inadmissible configuration not equivalent to Z.

First, suppose $m \geq 4$ & even $n \geq 6$. From Lemma 3, there exists a paired 2-DPC, made of s_1–t_1 and s_2–x paths, in G for every vertex $x \in (R_0 \cup R_{m-1}) \setminus \{s_1, t_1, s_2\}$such that $\{s_1, t_1, s_2, x\}$ is balanced and the four terminals in $\{s_1, t_1, s_2, x\}$ do not form an inadmissible configuration equivalent to A0, B0, or C0. Also, if $c(s_1) = c(t_1)$ and $s_2 \in R_0 \cup R_{m-1}$, then there exist two disjoint s_1–t_1 and s_2–x paths that cover all the vertices of G for $x = s_2$, because G is 1-fault Hamiltonian-laceable by Lemma 2. Inspecting each of the three inadmissible configurations each leads to the conclusion that two disjoint s_1–t_1 and s_2–x paths exist, provided $\{s_1, t_1, s_2, x\}$ is balanced, for every vertex x in one boundary row and at least one vertex x in the other boundary row, as required. Analogously, we can prove the theorem in each of the remaining three cases from Lemmas 4 through 6, and Lemma 2. Note that if the inadmissible configuration Z is not equal to F3 (where $m = 3$ & even $n \geq 6$), there exist required disjoint paths, s_1–t_1 and s_2–x paths, for every vertex x in one boundary row and at least one vertex x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced; otherwise, the required disjoint paths exist for every vertex x except one in one boundary row and at least two vertices x in the other boundary row such that $\{s_1, t_1, s_2, x\}$ is balanced. This completes the proof. □

REMARK 2. The number of such vertices x in Theorem 4 is at least $\frac{n}{2} + 1$.

THEOREM 5. *For distinct terminals $s_1, s_2, s_3 \in S$ and $t_1 \in T$ in an m × n cylindrical grid G with $m \geq 2$ and even $n \geq 4$ such that not all the four are of the same color, there exist vertices x and y in the boundary rows, possibly $x = s_2$ and/or $y = s_3$, such that G has three disjoint paths, s_1–t_1, s_2–x, and s_3–y paths, that altogether cover all the vertices of G.*

Proof. The proof will proceed by induction on m. Let $m = 2$ for the base step, where the two rows of G are both boundary ones. If $c(s_2) \neq c(s_3)$, then a Hamiltonian s_2–s_3 path exists in G since G is 1-fault Hamiltonian-laceable by Lemma 2. It suffices to divide the Hamiltonian path, represented as $\langle s_2, ..., x, s_1', ..., t_1', y, ..., s_3\rangle$, where $\{s_1', t_1'\} = \{s_1, t_1\}$, x is the predecessor of s_1', and y is the successor of t_1', into three subpaths: $\langle s_2,..., x\rangle$, $\langle s_1', ..., t_1'\rangle$, $\langle y,..., s_3\rangle$. If $c(s_2) = c(s_3)$, then $c(s_1) \neq c(s_2)$ or $c(t_1) \neq c(s_2)$, so we assume w.l.o.g. $c(s_1) \neq c(s_2)$. Then, there exists a Hamiltonian s_2–s_3 path in G–s_1 by Lemma 2. For a neighbor v of s_1 other than s_2 and s_3, the Hamiltonian path can be represented as $\langle s_2, ..., x, v', ..., t_1', y, ..., s_3\rangle$, where $\{v', t_1'\} = \{v_1, t_1\}$. It suffices to divide the Hamiltonian path into three subpaths, $\langle s_2, ..., x\rangle$, $\langle v', ..., t_1')$, $\langle y, ..., s_3\rangle$, and to combine the one-vertex path $\langle s_1\rangle$ with the second subpath through the edge (s_1, v).

Let $m \geq 3$ for the inductive step. We assume w.l.o.g. that R_0 contains no fewer terminals than R_{m-1}, i.e., $|R_0 \cap (S \cup T)| \geq |R_{m-1} \cap (S \cup T)|$. There are several possible cases depending on the distribution of terminals.

Case 1: *There is a boundary row that contains no terminal, i.e., $R_{m-1} \cap (S \cup T) = \emptyset$.* By the induction hypothesis, there are two vertices $x, y \in R_0 \cup R_{m-2}$ that admit three disjoint s_1–t_1, s_2–x, and s_3–y paths that cover all the vertices of the subgraph $G[R_{0,m-2}]$ induced by $R_{0,m-2}$. If exactly one of x and y is contained in R_{m-2}, say $x \in R_0$ and $y \in R_{m-2}$, it suffices to extend the s_3–y path to cover the vertices of R_{m-1}, i.e., concatenate the s_3–y path and a Hamiltonian w–y' path of the subgraph $G[R_{m-1}]$ induced by R_{m-1} for the neighbor $w \in R_{m-1}$ of y and a neighbor $y' \in R_{m-1}$ of w. If $x, y \in R_{m-2}$, then it suffices to extend the s_2–x and s_3–y paths to cover the vertices of R_{m-1}. That is, for the neighbor $u \in R_{m-1}$ of x and the neighbor $w \in R_{m-1}$ of y, we extract two disjoint u–x' and w–y' paths from a Hamiltonian cycle of $G[R_{m-1}]$, then concatenate the s_2–x and u–x' paths and concatenate again the s_3–y and w–y' paths.

Finally, suppose $x, y \notin R_{m-2}$, i.e., $x, y \in R_0$. If there is a nonterminal vertex v in R_{m-2}, i.e., $v \notin \{s_1, t_1, s_2, s_3\}$, then one of the three disjoint paths, $\{s_1$–t_1, s_2–$x\}$, and s_3–y paths, of $G[R_{0,m-2}]$ passes through v, hence passes through an edge (v, w) of $G[R_{m-2}]$. It suffices to reroute the path, instead of passing through the edge (v, w), to traverse a Hamiltonian v'–w' path of $G[R_{m-1}]$ for the neighbors v', $w' \in R_{m-1}$ of v and w, respectively. Now, let every vertex in R_{m-2} be a terminal, i.e., $R_{m-2} = \{s_1, t_1, s_2, s_3\}$ and $n = 4$. For the neighbors $s_1', t_1', s_2' \in R_{m-3}$, respectively, of s_1, t_1, and s_2, there are two disjoint s_1'–t_1' and s_2'–x paths for some $x \in R_0$ that cover $G[R_{0,m-3}]$ (The existence is by Theorem 4 if $m \geq 4$; the existence is obvious if $m = 3$). It suffices to concatenate the one-vertex path $<s_1>$, the $s_1' - t_1'$ path, and $<t_1>$ into an s_1–t_1 path, then concatenate again the one-vertex path $<s_2>$ and the $s_2' - x$ path, and extend $<s_3>$ to cover R_{m-1}.

Case 2: *There is a boundary row, say R_{m-1}, that contains a single terminal in $\{s_2, s_3\}$, say s_3, whose color is the same as at least one of the other terminals*. That is, $R_{m-1} \cap (S \cup T) = \{s_3\}$ and the three terminals $s_1, t_1, s_2 \in R_{0,m-2}$ are not of the same color. Then, for some $x \in R_0$, there exist disjoint s_1–t_1 and s_2–x paths that cover $G[R_{0,m-2}]$ by Theorem 4. It suffices to build a Hamiltonian s_3–y path of $G[R_{m-1}]$ for some y.

Case 3: $R_0 \cap (S \cup T) = \{s_1, s_2, s_3\}$. Assume w.l.o.g. that the three terminals in $\{s_1, t_1, s_2\}$ are not of the same color. It suffices to divide the Hamiltonian cycle $<s_1, \ldots, u, s_2, \ldots, x, s_3, \ldots, v>$ of $G[R_0]$ into three paths $<s_1, \ldots, u>$, $<s_2, \ldots, x>$, and $<s_3, \ldots, v>$, and then build two disjoint u'–t_1 and v'–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$, where $u', v' \in R_1$ are the neighbors of u and v, respectively. Note that $c(u') = c(s_2)$ and $c(v') = c(s_1)$, meaning that the three vertices of $\{u', v', t_1\}$ are not of the same color.

Case 4: $R_0 \cap (S \cup T) = \{s_1, t_1, s_2\}$. From the hypotheses of Cases 1 and 2, we can assume that $s_3 \in R_{m-1}$ and $c(s_1) = c(t_1) = c(s_2) \neq c(s_3)$. The proof is similar to that of Case 3. Dividing the Hamiltonian cycle $<s_1, \ldots, u, t_1, \ldots, v, s_2, \ldots, x>$ of $G[R_0]$ into $<s_1, \ldots, u>$, $<t_1, \ldots, v>$, and $<s_2, \ldots, x>$ paths and building two disjoin u'–v' and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$ leads to a requirement of three paths, where $u', v' \in R_1$ are the neighbors of u and v, respectively.

Case 5: $R_0 \cap (S \cup T) = \{s_2, s_3\}$. Similar to Case 3, assume w.l.o.g. that the three terminals in $\{s_1, t_1, s_2\}$ are not of the same color. It suffices to divide the Hamiltonian cycle of $G[R_0]$, represented as $<s_2, \ldots, x, s_3, \ldots, u>$ with (u, s_1), $(u, t_1) \notin E(G)$, into two paths $<s_2, \ldots, x>$ and $(s_3, \ldots, u)$, and then build two disjoint s_1–t_1 and u'– y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$, where $u' \in R_1$ is the neighbor of u (Note that R_1 contains at most one terminal from the hypothesis of Case 1).

Case 6: $R_0 \cap (S \cup T) = \{s_1, s_2\}$. Unless $c(s_1) \neq c(t_1) = c(s_2) = c(s_3)$, it suffices to divide the Hamiltonian cycle $<s_1, \ldots, u, s_2, \ldots, x>$ of $G[R_0]$, represented in a way that the neighbor $u' \in R_1$ of u is not a terminal, into s_1–u and s_2–x paths, and then build two disjoint u'–t_1 and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$. Suppose $c(s_1) \neq c(t_1) = c(s_2) = c(s_3)$ now. If $R_{m-1} \cap (S \cup T) = \{t_1, s_3\}$, then we can also build the three required paths symmetrically, so we assume that R_{m-1} contains a single terminal. If $(s_1, s_2) \in E(G)$, it suffices to divide the Hamiltonian cycle of $G[R_0]$ into $<s_2, x>$ and s_1–u paths for some $x, u \in R_0$, and then build two disjoint u'–t_1 and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$, where $u' \in R_1$ is the neighbor of u. If $(s_1, s_2) \notin E(G)$, it suffices to divide the Hamiltonian cycle $<s_1, \ldots, u, x, s_2, y, \ldots, v>$ of $G[R_0]$ into three paths $<s_1, \ldots, u>$, $<x, s_2>$, and $<y, \ldots, v>$, and then build a paired 2-DPC of $G[R_{1,m-1}]$, made of u'–t_1 and s_3–v' paths, where $u', v' \in R_1$ are the neighbors of u and v, respectively. The paired 2-DPC exists because R_{m-1} contains an odd number of terminals.

Case 7: $R_0 \cap (S \cup T) = \{s_1, t_1\}$. From the hypotheses of Cases 1, 2, and 5, we can assume that $R_{m-1} \cap (S \cup T) = \{s_3\}$ and $c(s_1) = c(t_1) = c(s_2) \neq c(s_3)$. From the Hamiltonian cycle of $G[R_0]$, we extract two disjoint paths, s_1–t_1 and u–v paths, that cover $G[R_0]$ for some $u, v \in R_0$, such that the neighbor $u' \in R_1$ of u is different from s_2. It suffices to build two disjoint s_2–u' and s_3–y paths that cover $G[R_{1,m-1}]$ for some $y \in R_{m-1}$.

Case 8: $R_0 \cap (S \cup T) = \{s_2\}$ *and* $R_{m-1} \cap (S \cup T) = \{s_3\}$. This case is reduced to Case 2.

Case 9: $R_0 \cap (S \cup T) = \{s_1\}$ *and* $R_{m-1} \cap (S \cup T) = \{s_3\}$. We assume $c(s_1) = c(t_1) = c(s_2) \neq c(s_3)$ from the hypothesis of Case 2. Let $t_1 \in R_i$ and $s_2 \in R_j$ for some $i, j \in \{1, \ldots, m-2\}$. If $i < j$, then for some edge (u, v) with $u \in R_i$, $v \in R_{i+1}$, and $c(u) = c(s_3)$, it suffices to build two disjoint s_1–t_1 and u–x paths that cover $G[R_{0,i}]$ for some $x \in R_0$, and build two disjoint s_2–v and s_3–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$. Analogously, if $j < i$, for some edge (u, v) with $u \in R_j$, $v \in R_{j+1}$, and $c(u) = c(s_3)$, we can build two disjoint s_1–u and s_2–x paths that cover $G[R_{0,j}]$ for some $x \in R_0$, and build two disjoint v–t_1 and s_3–y paths that cover $G[R_{j+1,m-1}]$ for some $y \in R_{m-1}$.

Finally, suppose $i = j$. Let s_1, t_1, and s_2, respectively, be contained in columns C_p, C_q, and C_r. Assume w.l.o.g. $q \leq p \leq r$ and $q = 0$.

Claim 1. There exist three disjoint s_1–t_1, s_2–u, and v–x paths that cover $G[R_{0,i}]$, where $u = v_1^i$, $v = v_{n-1}^i$, and $x = v_{p+1}^0$. Furthermore, each of the $\frac{n}{2} - 1$ edges (v_a^i, v_{a+1}^i) for odd $a \in \{1, 3, \ldots, n-3\}$ is visited by one of the three paths.

Proof of Claim 1. It holds true that $c(u) = c(v) = c(x) \neq c(s_1) = c(t_1) = c(s_2)$. If i is even, then $R_{0,i}$ has an odd number of rows, so possibly $p \in \{0, r\}$; if i is odd, then $R_{0,i}$ has an even number of rows, so $0 < p < r$ (Refer to Fig. 2). An s_1–t_1 path is obtained by concatenating a Hamiltonian s_1–v_0^{i-1} path of $G[R_{0,i-1} \cap C_{0,p}]$ and the one-vertex path $<t_1>$; set an s_2–u path to be $<v_r^i, v_{r-1}^i, \ldots, v_1^i>$;

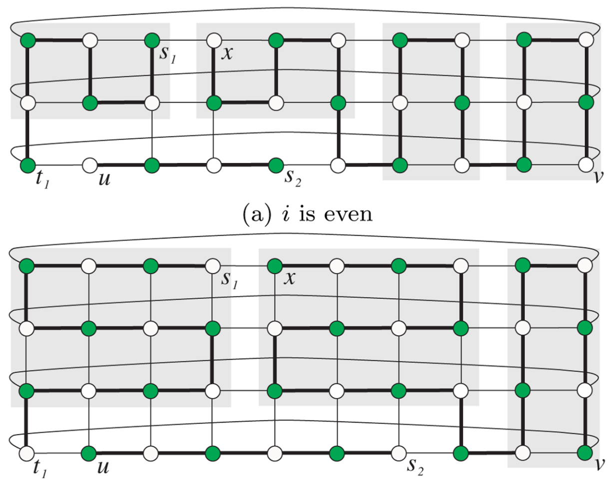


Fig. 2. Three disjoint s_1–t_1, s_2–u, and v–x paths in $G[R_{0,i}]$.

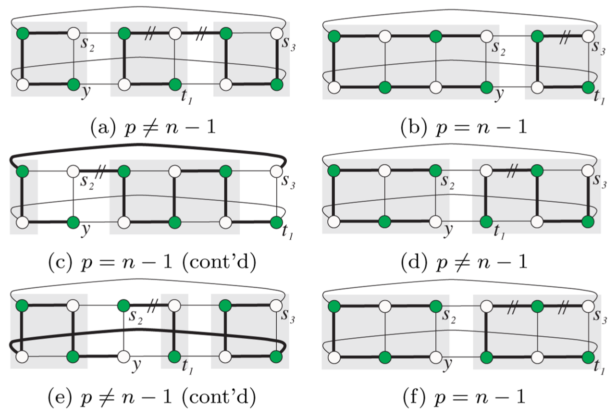


Fig. 3. Three disjoint u–t_1, s_a–v, and s_b–y paths that cover $G[R_{m-2,m-1}]$ for some $(a, b) \in \{(2, 3), (3, 2)\}$, where $c(s_2) = c(s_3)$ for (a), (b), and (c); $c(s_2) \neq c(s_3)$ for (d), (e), and (f).

in addition, a v–x path is obtained from concatenating a Hamiltonian $v^i_{n-1}-v^i_{n-2}$ path of $G[R_{0,i} \cap C_{n-2,n-1}]$, ..., a Hamiltonian $v^i_{r+3}-v^i_{r+2}$ path of $G[R_{0,i} \cap C_{r+2,r+3}]$, the one-vertex path $<v^i_{r+1}>$, and a Hamiltonian $v^{i-1}_{r+1}-v^0_{p+1}$ path of $G[R_{0,i-1} \cap C_{p+1,r+1}]$. The existence of the Hamiltonian paths in the induced subgraphs that are isomorphic to rectangular grids is due to Lemma 1(a). Thus, the claim is proven. □

Let u', $v' \in R_{i+1}$ be the neighbors of u and v, respectively. If $i \leq m-3$, it suffices to build two disjoint $u'-v'$ and s_3–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$, which exist by Theorem 4, and combine them with the three disjoint paths of Claim 1. So, let $i = m-2$ now, where $u' = v^{m-1}_1$, $v' = v^{m-1}_{n-1}$, and $s_3 = v^{m-1}_b$ for some even $b \in \{0, ..., n-2\}$ because $c(s_3) \neq c(u') = c(v')$. If $b = 0$, it suffices to set s_3–y and $u'-v'$ paths to be $<v^{m-1}_0>$ and $<v^{m-1}_1, v^{m-1}_2, ..., v^{m-1}_{n-1}>$, respectively, and combine the two with the three paths of Claim 1. If $b \geq 2$, we set an s_3–y path be $<v^{m-1}_b, v^{m-1}_{b-1}, ..., v^{m-1}_2>$ and set a $u'-v'$ path to be $<u', v^{m-1}_0, v'>$. To deal with the vertices $v^{m-1}_{b+1}, ..., v^{m-1}_{n-2}$ not visited until now, we use the fact shown in Claim 1 that every edge (v^i_a, v^i_{a+1}) for odd $a \in \{1, 3, ..., n-3\}$ is visited by one of the three disjoint paths of $G[R_{0,i}]$. To cover each pair of unvisited vertices v^{m-1}_c and v^{m-1}_{c+1} for odd $c \in \{b+1, ..., n-3\}$, it suffices to reroute the path that visits the edge $(v^{m-2}_c, v^{m-2}_{c+1})$ to traverse $<v^{m-2}_c, v^{m-1}_c, v^{m-1}_{c+1}, v^{m-2}_{c+1}>$.

Case 10: $R_0 \cap (S \cup T) = \{s_1\}$ *and* $R_{m-1} \cap (S \cup T) = \{t_1\}$. Let $s_2 \in R_i$ and $s_3 \in R_j$ for some $i, j \in \{1, ..., m-2\}$. Assume w.l.o.g. that the three terminals t_1, s_2, and s_3 are not of the same color. If $i < j$, we first pick up an edge (u, v) with $u \in R_i$ and $v \in R_{i+1}$ such that $c(u) \neq c(s_2)$ and $v \neq s_3$. Then, the three vertices of $\{s_1, s_2, u\}$ are not of the same color; also, the three vertices of $\{t_1, s_3, v\}$ are not of the same color because $c(v) = c(s_2)$. It suffices to build two disjoint s_1–u and s_2–x paths that cover $G[R_{0,i}]$ for some $x \in R_0$, and combine them with the two disjoint v–t_1 and s_3–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$. The case where $j < i$ is symmetric to the case where $i < j$, so we consider the remaining case where $i = j$ hereafter.

Claim 2. There exists an edge (u, v) of $G[R_i]$ with $(v, s_1) \notin E(G)$ such that for some $y \in R_{m-1}$, the subgraph $G[R_{i,m-1}]$ contains three disjoint paths, composed of either u–t_1, s_2–v, and s_3–y paths, or u–t_1, s_3–v, and s_2–y paths, that cover all the vertices of $G[R_{i,m-1}]$.

Proof of Claim 2. If $i \leq m-3$, then $G[R_{i,m-1}]$ contains three or more rows. For an edge (u, v) of $G[R_i]$ with $u \in \{s_2, s_3\}$ and $(v, s_1) \notin E(G)$, it suffices to decompose the Hamiltonian cycle of $G[R_i]$, represented as $<u, ..., w, s_3, ..., z, s_2, ..., v>$, into three paths $<u, ..., w>$, $<s_3, ..., z>$, and $<s_2, ..., v>$, and then build disjoint w'–t_1 and z'–y paths that cover $G[R_{i+1,m-1}]$ for some $y \in R_{m-1}$, where w', $z' \in R_{i+1}$ are the neighbors of w and z, respectively. Note that $c(w') = c(s_3)$ and $c(z') = c(s_2)$, so the vertices of $\{t_1, w', z'\}$ are not of the same color. Now, suppose $i = m-2$, where $G[R_{i,m-1}]$ contains exactly two rows. Let $t_1 \in C_p$, $s_2 \in C_q$, and $s_3 \in C_r$ for some $p, q, r \in \{0, ..., n-1\}$.

For the first case, suppose $c(s_2) = c(s_3)$, so $c(s_2) = c(s_3) \neq c(t_1)$ from our assumption. We further assume w.l.o.g. that $q < p \leq r$ and $r = n-1$ (See Fig. 3(a)–(c)). If $p \neq n-1$, it suffices to set an s_2–y path to be a Hamiltonian $s_2-v^{m-1}_q$ path of $G[R_{m-2,m-1} \cap C_{0,q}]$, and then decompose the s_3–t_1 path, built by concatenating a Hamiltonian $s_3-v^{m-2}_{p+1}$ path of $G[R_{m-2,m-1} \cap C_{p+1,n-1}]$ and a Hamiltonian $v^{m-2}_p-t_1$ path of $G[R_{m-2,m-1} \cap C_{q+1,p}]$, by deleting an edge $(u, v) = (v^{m-2}_{p-1}, v^{m-2}_p)$ or $(v^{m-2}_p, v^{m-2}_{p+1})$ so that $(v, s_1) \notin E(G)$. If $p = n-1$, the required three paths are obtained in one of the following two ways: (i) set an s_2–y path to be a Hamiltonian $s_2-v^{m-1}_q$ path of $G[R_{m-2,m-1} \cap C_{0,q}]$, and then decompose the Hamiltonian s_3–t_1 path of $G[R_{m-2,m-1} \cap C_{q+1,n-1}]$ through $(u, v) = (v_{n-2}^{m-2}, v_{n-1}^{m-2})$; or (ii) concatenate $<s_3>$, a Hamiltonian $v_0^{m-2}-v_{q-1}^{m-1}$ path of $G[R_{m-2,m-1} \cap C_{0,q-1}]$, and $<v_q^{m-1}>$ into an s_3–y path, and then decompose the s_2–t_1 path, built by concatenating $<s_2>$, a Hamiltonian $v_{q+1}^{m-2}-v_{n-2}^{m-1}$ path of $G[R_{m-2,m-1} \cap C_{q+1,n-2}]$ and $<t_1>$, through $(u, v) = (v_{q+1}^{m-2}, v_q^{m-2})$.

For the second case, suppose $c(s_2) \neq c(s_3)$. Assume w.l.o.g. that $c(t_1) = c(s_2) \neq c(s_3)$ and moreover, $q < p \leq r = n-1$ (See Fig. 3(d)–(f)). If $p \neq n-1$, the three required paths are obtained in one of the following two ways: (i) set an s_2–y path to be a Hamiltonian $s_2-v_{p-1}^{m-1}$ path of $G[R_{m-2,m-1} \cap C_{0,p-1}]$, and then decompose the Hamiltonian s_3–t_1 path of $G[R_{m-2,m-1} \cap C_{p,n-1}]$ through $(u, v) = (v_p^{m-2}, v_{p+1}^{m-2})$; or (ii) concatenate a Hamiltonian $s_3-v_{q-1}^{m-1}$ path of $G[R_{m-2,m-1} \cap (C_{0,q-1} \cup C_{p+1,n-1})]$ and $<v_q^{m-1}>$ into an s_3–y path, and then decompose the s_2–t_1 path, built by concatenating $<s_2>$ and a Hamiltonian $v_{q+1}^{m-2} - t_1$ path of $G[R_{m-2,m-1} \cap C_{q+1,p}]$, through $(u, v) = (v_{q+1}^{m-2}, v_q^{m-2})$. If $p = n-1$, assuming w.l.o.g. $q \neq n-2$, it suffices to set an s_2–y path to be a Hamiltonian $s_2-v_q^{m-1}$ path of $G[R_{m-2,m-1} \cap C_{0,q}]$ and then decompose the Hamiltonian s_3–t_1 path of $G[R_{m-2,m-1} \cap C_{q+1,n-1}]$ through an edge $(u, v) = (v_{n-3}^{m-2}, v_{n-2}^{m-2})$ or $(v_{n-2}^{m-2}, v_{n-1}^{m-2})$. Thus, the claim is proven. □

Let $u', v' \in R_{i-1}$ be the neighbors of u and v, respectively. Two disjoint s_1–u' and v'–x paths that cover $G[R_{0,i-1}]$ for some $x \in R_0$ remain to be built. If $i \geq 2$, the two disjoint paths exist by Theorem 4; if $i = 1$, dividing the Hamiltonian cycle $<s_1, ..., u', v', ..., x>$, where $v' \neq s_1$, of $G[R_0]$ results in two paths $<s_1, ..., u'>$ and $<v', ..., x>$, as required. If we combine the two paths of $G[R_{0,i-1}]$ with the three paths of Claim 2, we obtain the required three paths that cover G. This completes the entire proof. □

REMARK 3. If distinct terminals $s_1, s_2 \in S$ and $t_1, t_2 \in T$ (instead of $s_1, s_2, s_3 \in S$ and $t_1 \in T$) are given in an $m \times n$ cylindrical grid with $m \geq 2$ and even $n \geq 4$, then there exist three disjoint paths, s_1–t_1, s_2–x, and t_2–y paths (instead of s_1–t_1, s_2–x, and s_3–y paths), that altogether cover all the vertices.

IV. PAIRED 3-DPC IN BIPARTITE TOROIDAL GRIDS

In this section, we will show that every $m \times n$ bipartite toroidal grid with $(m, n) \neq (4, 4)$ has a paired 3-DPC joining S and T for any disjoint source and sink sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ such that $S \cup T$ is balanced. The 6 × 4 and 6 × 6 toroidal grids admit a paired 3-DPC joining S and T for any such terminal sets S and T, while the 4 × 4 toroidal grid does not, as shown in Fig. 4. Lemma 7 below was verified from a computer program that exhaustively searches for DPCs. The source code may be downloaded from http://tcs.catholic.ac.kr/~jhpark/papers/toroidal_grid.zip.

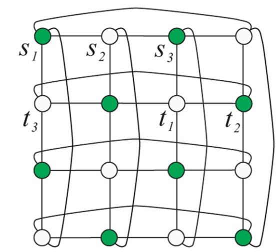


Fig. 4. A conguration that does not admit a paired 3-DPC. Every s_i–t_i path that does not pass through a terminal as an intermediate vertex contains at least 6 vertices, whereas the toroidal grid has fewer than 18 vertices.

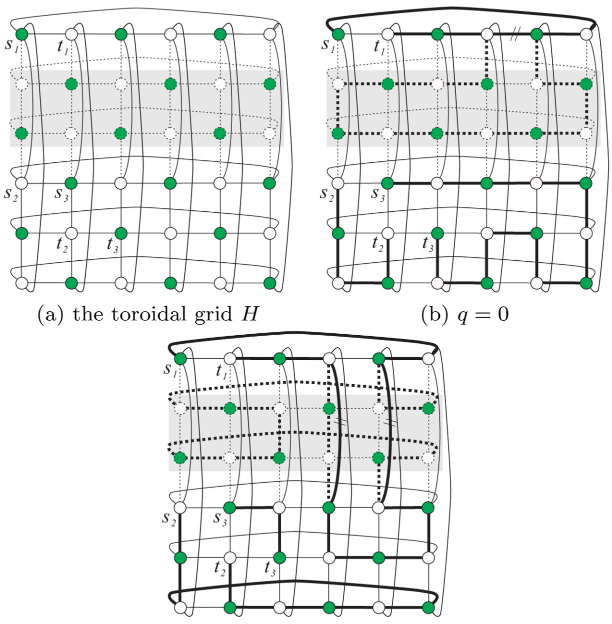
LEMMA 7. *Let G be a* 6 × 4 *or* 6 × 6 *toroidal grid, in which disjoint source and sink sets* $S = \{s_1, s_2, s_3\}$ *and* $T = \{t_1, t_2, t_3\}$ *are given. Then, G has a paired* 3*-DPC joining S and T if* $S \cup T$ *is balanced.*

One of the natural approaches would be the reduction of our problem to a problem on a smaller bipartite toroidal grid. This is possible if there are two consecutive rows that contain no terminal as follows:

LEMMA 8 (Row reduction). *An* $m \times n$ *bipartite toroidal grid G with* $m \geq 6$ *has a paired* 3*-DPC joining* $S = \{s_1, s_2, s_3\}$ *and* $T = \{t_1, t_2, t_3\}$ *if (i)* $S \cup T$ *is balanced, (ii) there are two consecutive rows* R_p *and* R_{p+1} *that contain no terminal, and (iii) an* $(m-2) \times n$ *toroidal grid has a paired* 3*-DPC joining S′ and T′ for any disjoint terminal sets S′ and T′ such that* $S' \cup T'$ *is balanced.*

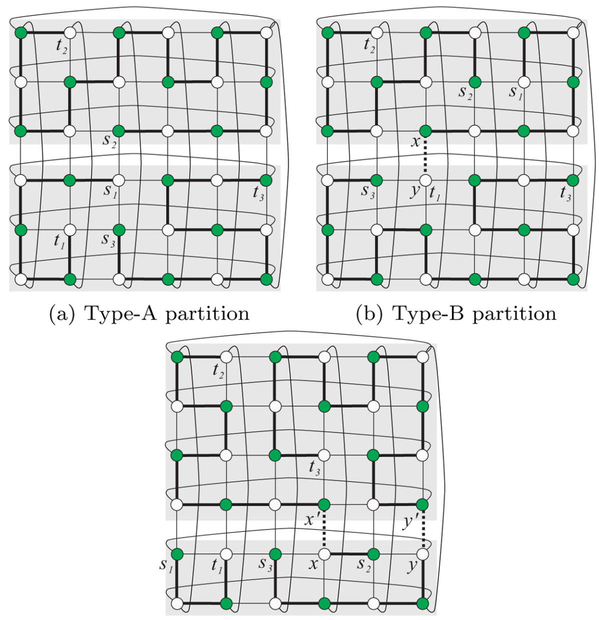
Proof. Let H denote the $(m-2) \times n$ toroidal grid, obtained from G by deleting the vertices of $R_{p,p+1}$ and adding *n virtual* edges (v_j^{p-1}, v_j^{p+2}) for $j \in \{0, ..., n-1\}$, as shown in Fig. 5(a). Then, by hypothesis (iii) of the lemma, H has a paired 3-DPC joining S and T. If none of the virtual edges is passed through by a path in the 3-DPC of H (see Fig. 5(b)), then for an edge in row $p-1$ or $p+2$, say $(v_j^{p-1}, v_{j+1}^{p-1})$ w.l.o.g., that is covered by the 3-DPC of H, replacing the edge with a path obtained by concatenating $<v_j^{p-1}>$, a Hamiltonian $v_j^p-v_{j+1}^p$ path of $G[R_{p,p+1}]$, and $<v_{j+1}^{p-1}>$ results in a paired 3-DPC of G. Now, suppose that there is a virtual edge that is covered by the 3-DPC of H (see Fig. 5(c)). Let $\{(v_j^{p-1}, v_j^{p+2}): j \in \{j_1, ..., j_q\}\}$ be the set of such virtual edges, and assume $j_1 < \cdots < j_q$. A paired 3-DPC of G can be built by replacing the virtual edge $(v_{j_i}^{p-1}, v_{j_i}^{p+2})$ with a path obtained by concatenating $<v_{j_i}^{p-1}>$, a Hamiltonian $v_{j_i}^p-v_{j_i}^{p+1}$ path of $G[R_{p,p+1} \cap C_{j_i,j_{i+1}-1}]$, and $<v_{j_i}^{p+2}>$ if $i < q$; with a path obtained by concatenating $<v_{j_q}^{p-1}>$, a Hamiltonian $v_{j_q}^p-v_{j_q}^{p+1}$ path of $G[R_{p,p+1} \cap (C_{j_q,n-1} \cup C_{0,j_1-1})]$, and $<v_{j_q}^{p+2}>$ if $i = q$. Thus, the lemma is proven. □

An $m \times n$ bipartite toroidal grid with $m \geq 6$ is said to be *row-reducible* if there are two consecutive rows R_p and



(a) the toroidal grid H (b) $q = 0$ (c) $j_1 = 3$ and $j_2 = 4$

Fig. 5. Illustrations of the row reduction, where $R_{1,2}$ contains no terminal.



(a) Type-A partition (b) Type-B partition (c) Type-C partition

Fig. 6. Three types of partitions of a toroidal grid into two cylindrical grids.

R_{p+1} that contain no terminals. Besides the row reduction of Lemma 8, we can try a partition of the $m \times n$ toroidal grid into two cylindrical grids, each having at least two rows, so as to build a paired 3-DPC in the toroidal grid. Three types of such partitions are investigated in Lemmas 9, 10, and 11 below and illustrated in Fig. 6.

LEMMA 9 (Type-A partition). *An $m \times n$ bipartite toroidal grid G has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced and there are r, $2 \le r \le m - 2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain four terminals s_a, t_a, s_b, and t_b for some $a, b \in \{1, 2, 3\}$ in total such that the subgraph $G[R_{p,p+r-1}]$ induced by $R_{p,p+r-1}$ has a paired 2-DPC composed of s_a–t_a and s_b–t_b paths.*

Proof. The subgraph $G - R_{p,p+r-1}$ contains two terminals s_c and t_c with $c(s_c) \neq c(t_c)$, so there exists a Hamiltonian s_c–t_c path in the subgraph by Lemma 2. A paired 2-DPC of $G[R_{p,p+r-1}]$ along with the Hamiltonian path form a paired 3-DPC of G. □

LEMMA 10 (Type-B partition). *An $m \times n$ bipartite toroidal grid G has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced and there are r, $2 \le r \le m - 2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain three terminals s_a, t_a, and s_b, for some $a, b \in \{1, 2, 3\}$ in total such that the three are not of the same color.*

Proof. In the subgraph $G[R_{p,p+r-1}]$, there are two disjoint s_a–t_a and s_b–x paths for some $x \in R_p \cup R_{p+r-1}$ that cover all the vertices of the subgraph; moreover, the number of such vertices x is at least $\frac{n}{2} + 1$ by Theorem 4. Consider the subgraph H of G induced by $R_{0,p-1} \cup R_{p+r,n-1}$ now (i.e., $H = G - R_{p+r,n-1}$), in which there are three terminals s_c, t_c, and t_b for some $c \in \{1, 2, 3\}$ with $c \neq a, b$. Also, the three terminals of H are not of the same color, so there exist two disjoint $s_c - t_c$ and $t_b - y$ paths that cover H for at least $\frac{n}{2} + 1$ choices of $y \in R_{p-1} \cup R_{p+r}$ by Theorem 4 again. It follows that there is an edge (x, y) of G, where $x \in R_p \cup R_{p+r-1}$ and $y \in R_{p-1} \cup R_{p+r}$, that admits not only a 2-DPC, made of s_a–t_a and s_b–x paths, of $G[R_{p,p+r-1}]$ but also a 2-DPC, made of s_c–t_c and t_b–y paths, of H, because $c(x) \neq c(y)$ and there are at least $\frac{n}{2} + 1$ choices of each of x and y. It suffices to combine the s_b–x path with the t_b–y path into an s_b–t_b path through the edge (x, y), completing the proof. □

LEMMA 11 (Type-C partition). *An $m \times n$ bipartite toroidal grid G has a paired 3-DPC joining $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ if $S \cup T$ is balanced, G is not row-reducible, and there are r, $2 \le r \le m - 2$, consecutive rows $R_p, \ldots, R_{p+r-1}$ that contain two terminals α and β in total such that*

- $c(\alpha) = c(\beta)$ or $\alpha, \beta \notin R_p \cup R_{p+r-1}$ *when $r \ge 4$,*
- $c(\alpha) = c(\beta)$ & $|\{\alpha, \beta\} \cap R_{p+1}| = 1$
  or $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \in \mathcal{K}$ & $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+2}| = 1$,
  or $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \notin \mathcal{K}$ & $\alpha, \beta \in R_{p+1}$
  or $c(\alpha) \neq c(\beta)$ & $(\alpha, \beta) \notin \mathcal{K}$ & $\alpha, \beta \in R_{p+1}$ & $(\alpha, \beta) \notin E(G)$ *when $r = 3$,*

• $c(\alpha) = c(\beta)$ & $(\alpha, \beta) \notin \mathcal{K}$ & $|\{\alpha, \beta\} \cap R_p| = 1$ *when* $r = 2$,

where $\mathcal{K} = \{(s_1, t_1), (s_2, t_2), (s_3, t_3)\}$.

Proof. Let H be the subgraph $G - R_{p,p+r-1}$ induced by $R_{0,p-1} \cup R_{p+r,n-1}$, in which there are four terminals, say s_a, t_a, α', and β' for some $a \in \{1, 2, 3\}$, so that $S \cup T = \{s_a, t_a, \alpha, \alpha', \beta, \beta'\}$, where (α, α'), $(\beta, \beta') \in \mathcal{K}$, or (α, β), $(\alpha', \beta') \in \mathcal{K}$, or (α, β'), $(\alpha', \beta) \in \mathcal{K}$. The four terminals of H are not of the same color since $S \cup T$ is balanced. So, from Theorem 5, there exist three disjoint s_a–t_a, α'–x, and β'–y paths that cover H for some $x, y \in R_{p-1} \cup R_{p+r}$.

Let x', $y' \in R_p \cup R_{p+r-1}$ be the neighbors of x and y, respectively.

CLAIM 3. For the two terminals α and β of $G[R_{p,p+r-1}]$ satisfying the hypothesis of the lemma, (i) $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$; moreover, (ii) $G[R_{p,p+r-1}]$ has three kinds of paired 2-DPCs, a DPC made of α–x' and β–y' paths, a DPC made of α–y' and β–x' paths, and a DPC made of α–β and x'–y' paths.

Proof of Claim 3. Within the scope of this proof, x' and y' as well as α and β are said to be terminals. Observing that $\{\alpha, \beta, x', y'\}$ is balanced, we prove the assertion (i) first. If $c(\alpha) = c(\beta)$, then $c(x') = c(y') \neq c(\alpha) = c(\beta)$, so $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$; if $\alpha, \beta \notin R_p \cup R_{p+r-1}$, then $\{x', y'\} \cap \{\alpha, \beta\} = \emptyset$ obviously. Inspecting the hypothesis of the lemma leads to $c(\alpha) = c(\beta)$ or $\alpha, \beta \notin R_p \cup R_{p+r-1}$, proving (i). For the proof of the assertion (ii), let $\alpha \in R_i$ and $\beta \in R_j$ for some $i, j \in \{p, \ldots, p + r - 1\}$. First, let $r \geq 4$. It follows that $i \neq j$ and $\{i, j\} \neq \{p, p + r - 1\}$; suppose otherwise, G would be row-reducible. This leads to the conclusion that there is a (non-boundary) row that contains a single terminal, meaning the required 2-DPCs exist by Lemmas 3 and 4 (also, by Remark 1). Secondly, let $r = 3$. If $|\{\alpha, \beta\} \cap R_{p+1}| = 1$, then R_{p+1} contains a single terminal, so the required 2-DPCs exist. If $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+2}| = 1$, $c(\alpha) = c(\beta)$, and $(\alpha, \beta) \in \mathcal{K}$, then the four terminals in $\{\alpha, \beta, x', y'\}$ cannot form an inadmissible configuration of Lemmas 4 and 6, so the required 2-DPCs exist. Analogously, we can see that the required 2-DPCs exist for the remaining two cases where $\alpha, \beta \in R_{p+1}$. Finally, let $r = 2$. If $c(\alpha) = c(\beta)$, $(\alpha, \beta) \in \mathcal{K}$, and $i \neq j$ (i.e., $|\{\alpha, \beta\} \cap R_p| = |\{\alpha, \beta\} \cap R_{p+1}| = 1$), then the four terminals in $\{\alpha, \beta, x', y'\}$ cannot form an inadmissible configuration of Lemmas 4 and 5, so the required 2-DPCs exist. Thus, the claim is proven. □

Combining the α'–x and β'–y paths of H with one of the three paired 2-DPCs of $G[R_{p,p+r-1}]$ through the edges (x, x') and (y, y') leads to a paired 3-DPC of G, as required. This completes the proof. □

Now, we are ready to prove our main theorem.

THEOREM 6. *An* $m \times n$ *bipartite toroidal grid* G *with* $(m, n) \neq (4, 4)$ *has a paired* 3*-DPC joining disjoint terminal sets* $S = \{s_1, s_2, s_3\}$ *and* $T = \{t_1, t_2, t_3\}$ *if and only if* $S \cup T$ *is balanced.*

Proof. The necessity part is straightforward from the fact that the two color classes of G are always the same in size. The sufficiency proof will proceed by induction on $m + n$, where m and n are both even integers with $m, n \geq 4$ and $m + n \geq 10$. Assume w.l.o.g. $m \geq n$. The base step of $(m, n) = (6, 4)$ is due to Lemma 7. Moreover, the theorem holds true for the case of $(m, n) = (6, 6)$ by Lemma 7 again, so we assume $m \geq 8$ for the inductive step. Keep in mind that if G is row-reducible, then G has a paired 3-DPC joining S and T by Lemma 8 because by the induction hypothesis, an $(m - 2) \times n$ bipartite toroidal grid has a paired 3-DPC joining any disjoint terminal sets S' and T' of size 3 each such that $S' \cup T'$ is balanced. We assume w.l.o.g. that R_0 contains as many terminals as the other rows, i.e., $|R_0 \cap (S \cup T)| \geq |R_i \cap (S \cup T)|$ for all $i \in \{1, ..., m-1\}$. There are three cases according to the size of $R_0 \cap (S \cup T)$.

Case 1: $|R_0 \cap (S \cup T)| \geq 3$. The $m - 1$ (≥ 7) rows other than R_0 contain 3 or fewer terminals in total, so (i) G is row-reducible, or (ii) $m = 8$ and each of the three rows R_2, R_4, and R_6 contains a single terminal. For possibility (i), G has a paired 3-DPC joining S and T by the induction hypothesis and Lemma 8; for possibility (ii), G admits a type-C partition w.r.t. $R_{1,5}$, and hence G has a paired 3-DPC joining S and T by Lemma 11.

Case 2: $|R_0 \cap (S \cup T)| = 2$.

Case 2.1: $|R_i \cap (S \cup T)| = 2$ *for some* $i \in \{1, ..., m-1\}$. In this case, there are at most three rows other than R_0, each of which contains a terminal. It follows that G is row-reducible, or $m = 8$ and the three rows R_2, R_4, and R_6 each contains a terminal. If G is row-reducible, we are done by the induction hypothesis and Lemma 8. If $i = 2$, i.e., R_2 contains two terminals, then G has a paired 3-DPC joining S and T by Lemma 11 because G admits a type-C partition w.r.t. $R_{3,7}$; symmetrically in the case of $i = 6$, G is also type-C-partitionable. Let $i = 4$ now. There are two possibilities: (i) $R_0 \cap (S \cup T) = \{s_a, t_a\}$ for some a, and (ii) $R_0 \cap (S \cup T) \neq \{s_a, t_a\}$ for all a.

For the first possibility, suppose s_a, $t_a \in R_0$. If $c(s_a) \neq c(t_a)$, then G admits a type-A partition w.r.t. $R_{2,7}$, hence G has a required 3-DPC by Lemma 9 (Note that the four terminals in $(S \cup T) \setminus \{s_a, t_a\}$ do not form an inadmissible configuration in the induced subgraph $G[R_{2,7}]$ since there is a row, say R_2, that contains an odd number of terminals). If $c(s_a) = c(t_a)$, then there is a terminal α in R_2 or in R_6 such that $c(\alpha) \neq c(s_a) = c(t_a)$, hence, assuming w.l.o.g. $\alpha \in R_2$, G admits a type-B partition w.r.t. $R_{0,2}$ and has a required 3-DPC by Lemma 10.

For the second possibility, suppose s_a, $s_b \in R_0$ for some a, $b \in \{1, 2, 3\}$ with $a \neq b$ (or symmetrically, s_a, $t_b \in R_0$).

For the two terminals, denoted α and β, in R_4, if $\{\alpha, \beta\} = \{s_c, t_c\}$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$, then a paired 3-DPC can be constructed in a way symmetric to the first possibility where $s_a, t_a \in R_0$. So, we assume $\{\alpha, \beta\} \neq \{s_c, t_c\}$. If either $c(s_a) = c(s_b)$ or $c(s_a) \neq c(s_b)$ & $(s_a, s_b) \notin E(G)$, then G admits a type-C partition w.r.t. $R_7 \cup R_{0,1}$, hence G has a required 3-DPC by Lemma 11. Similarly, if either $c(\alpha) = c(\beta)$ or $c(\alpha) \neq c(\beta)$ & $(\alpha, \beta) \notin E(G)$, then G is type-C-partitionable w.r.t. $R_{3,5}$ and has a required 3-DPC. So, we further assume $(s_a, s_b), (\alpha, \beta) \in E(G)$ ($c(s_a) \neq c(s_b)$ and $c(\alpha) \neq c(\beta)$). If $t_a \in R_2$ or $t_b \in R_2$, then G is type-B-partitionable w.r.t. $R_{0,2}$ and thus G has a required 3-DPC by Lemma 10; also, G is type-B-partitionable w.r.t. $R_{6,7} \cup R_0$ if $t_a \in R_6$ or $t_b \in R_6$.

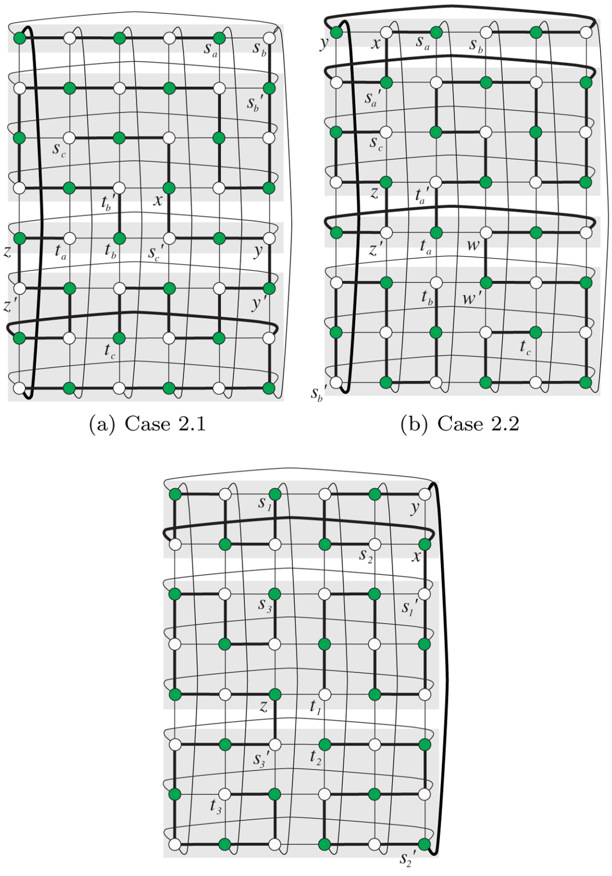
Finally, there remains a case where $t_a, t_b \in R_4$ and $s_c, t_c \in R_2 \cup R_6$, say $s_c \in R_2$ and $t_c \in R_6$, and moreover $(s_a, s_b), (t_a, t_b) \in E(G)$ and $c(s_c) \neq c(t_c)$. None of the three types of a partition can be applied in this case, so we will devise a direct construction of a paired 3-DPC joining S and T. We assume w.l.o.g. that $c(s_b) = c(s_c)$, $s_a = v^0_{n-2}$, and $s_b = v^0_{n-1}$, and let $t_b = v^4_j$ for some j. The construction will be completed in five steps as follows (see Fig. 7(a)):

1: Find a Hamiltonian $s_a - v^0_0$ path, $\langle v^0_{n-2}, ..., v^0_0 \rangle$, in $G[R_0] - s_b$.
2: Let $x = v^3_{j+1}$ if $t_a \neq v^4_{j+1}$; let $x = v^3_{j-1}$ otherwise. For $s_b' = v^1_{n-1}$ and $t_b' = v^3_j$, find a paired 2-DPC composed of s_b'–t_b' and s_c–x paths in $G[R_{1,3}]$.
3: Let s_c' be the neighbor of x in R_4. Divide the Hamiltonian s_c'–t_a path of $G[R_4] - t_b$ into s_c'–y and z–t_a paths, by deleting an arbitrary edge (y, z) of the Hamiltonian path.
4: Let y' and z' be the respective neighbors of y and z in R_5. Find a paired 2-DPC composed of $y'-t_c$ and v^7_0-z' paths in $G[R_{5,7}]$.
5: Concatenating the s_a–v^0_0, v^7_0-z', and z–t_a paths results in an s_a–t_a path; concatenating the one vertex path $\langle s_b \rangle$, the s_b'–t_b' path, and $\langle t_b \rangle$ leads to an s_b–t_b path; finally, concatenating the s_c–x, s_c'–y, and $y'-t_c$ paths leads to an s_c–t_c path.

The paired 2-DPCs in Steps 2 and 4 exist due to Lemmas 4 and 6 (also, due to Remark 1).

Case 2.2: *$|R_i \cap (S \cup T)| \leq 1$ for all $i \in \{1, ..., m-1\}$.* There are exactly four rows other than R_0, each of which contains a terminal, so G is row-reducible (and we are done) or $m \leq 10$. If $m = 10$, then each of the four rows R_2, R_4, R_6, and R_8 contains a single terminal, hence G admits a type-C partition w.r.t. $R_{1,5}$ and has a required 3-DPC by Lemma 11. Suppose $m = 8$ hereafter. Let r be the maximum number of consecutive rows, including R_0, each of which contains a terminal; also, let R_p, ..., R_q denote the remaining $8 - r$ consecutive rows (Note that $R_{p,q}$ contains $5 - r$ terminals; but R_p and R_q contain no terminal). It follows that $r \leq 3$ because G is not row-reducible. If $r = 3$, then each of R_{p+1} and R_{p+3} contains a single terminal, hence G admits a type-C-partition w.r.t. $R_{p,p+4}$ and has a required 3-DPC. If $r = 2$, then each of R_{p+1} and R_{p+4} contains a single terminal; also, either R_{p+2} or R_{p+3} contains a single terminal. This leads to the conclusion that G is type-C-partitionable (w.r.t. $R_{p,p+3}$ for the former case and w.r.t. $R_{p+2,p+5}$ for the latter case) and has a required 3-DPC. Finally, if $r = 1$, then each of R_2 and R_6 contains a single terminal; also, two of the three R_3, R_4, and R_5 contain a single terminal. If each of R_3 and R_5 contains a single terminal (but R_4 does not), then G is type-C-partitionable w.r.t. $R_{1,4}$. So, we assume w.l.o.g. each of R_4 and R_5 contains a single terminal, i.e., $|R_j \cap (S \cup T)| = 1$ for $j \in \{2, 4, 5, 6\}$.

Let α and β denote the two terminals in R_0. First, suppose $c(\alpha) = c(\beta)$. If $\{\alpha, \beta\} = \{s_a, t_a\}$ for some $a \in \{1, 2, 3\}$, then assuming w.l.o.g. that the terminal in R_2 has a color different from $c(\alpha)$, G is type-B-partitionable w.r.t. $R_{0,2}$. If $\{\alpha, \beta\} \neq \{s_a, t_a\}$ for all a, then G is type-C-partitionable w.r.t. $R_7 \cup R_{0,1}$. Secondly, suppose $c(\alpha) \neq$



(a) Case 2.1 (b) Case 2.2

(c) Case 3

Fig. 7. Illustrations of the proof of Theorem 6 for the cases to which none of the three types of a partition is applicable.

$c(\beta)$. If $\{\alpha, \beta\} = \{s_a, t_a\}$ for some a, then G is type-A-partitionable w.r.t. $R_{2,7}$. If $\{\alpha, \beta\} \neq \{s_a, t_a\}$ for all a, and moreover $(\alpha, \beta) \notin E(G)$, then G is type-C-partitionable w.r.t. $R_7 \cup R_{0,1}$. So, we further assume $\{\alpha, \beta\} = \{s_a, s_b\}$ for some $a, b \in \{1, 2, 3\}$ with $a \neq b$, and $(s_a, s_b) \in E(G)$. If R_2 contains t_a or t_b, then G is type-B-partitionable w.r.t. $R_{0,2}$; if R_6 contains t_a or t_b, then G is also type-B-partitionable w.r.t. $R_{6,7} \cup R_0$. There remains a case where $(R_2 \cup R_6) \cap (S \cup T) = \{s_c, t_c\}$ for some $c \in \{1, 2, 3\}$ with $c \neq a, b$. Assume w.l.o.g. $s_c \in R_2$ and $t_c \in R_6$, and moreover $t_a \in R_4$ and $t_b \in R_5$. If $c(t_a) = c(t_b)$, then G is type-C-partitionable w.r.t. $R_{3,5}$; also, if $c(t_b) = c(t_c)$, then G is type-C-partitionable w.r.t. $R_{5,7}$. Under the condition $c(t_a) = c(t_c) \neq c(t_b) = c(s_c)$, we give a direct construction of a paired 3-DPC below for the remaining case (see Fig. 7(b)).

1: Find a Hamiltonian s_a–s_b path in $G[R_0]$. Let the Hamiltonian path be represented as $<s_a, \ldots, x, y, \ldots, s_b>$, possibly $x = s_a$, for some x with $c(x) = c(s_c)$.
2: For the neighbor $s_a' \in R_3$ of x, the neighbor $t_a' \in R_3$ of t_a, and a neighbor $z \in R_1$ of t_a', find a paired 2-DPC made of s_a'–t_a' and s_c–z paths in $G[R_{1,3}]$.
3: For the neighbor $z' \in R_4$ of z and the neighbor $w' \in R_4$ of t_a other than z', find a Hamiltonian z'–w path in $G[R_4] - t_a$.
4: For the neighbor $s_b' \in R_7$ of y and the neighbor $w' \in R_5$ of w, find a paired 2-DPC composed of s_b'–t_b and w'–t_c paths in $G[R_{5,7}]$.
5: Concatenating the s_a–x path, the s_a'–t_a' path, and $<t_a>$ results in an s_a–t_a path; concatenating the s_b–y and s_b'–t_b paths leads to an s_b–t_b path; finally, concatenating the s_c–z, z'–w, and w'–t_c paths leads to an s_c–t_c path.

Case 3: $|R_0 \cap (S \cup T)| = 1$. Let r denote the maximum number of consecutive rows where each of which contains a terminal; assume w.l.o.g. that $R_0, \ldots, R_{r-1}$ are such consecutive rows. First, suppose $r = 1$. Then, G is type-C-partitionable w.r.t. $R_{m-1} \cup R_{0,q+1}$ for some $q \geq 1$ such that R_q contains a terminal but R_j does not for all $j \in \{1, \ldots, q-1\}$. Secondly, suppose $r = 2$. Then, G is also type-C-partitionable w.r.t. $R_{m-1} \cup R_{0,2}$. Thirdly, suppose $r = 3$. Then, G is row-reducible or $m \leq 10$. If $m = 10$, then each of R_4, R_6, and R_8 contains a single terminal, so G is type-C-partitionable w.r.t. $R_{3,7}$. Let $m = 8$ now. The rows R_3 and R_7 contain no terminal, so each of R_4, R_5, R_6 contains a terminal, i.e., $|R_j \cap (S \cup T)| = 1$ iff $j \in \{0, 1, 2, 4, 5, 6\}$. Let α_i denote the terminal in R_i. If $c(\alpha_0) = c(\alpha_1)$, then G is type-C-partitionable; if $c(\alpha_1) = c(\alpha_2)$, then G is also type-C-partitionable; so, $c(\alpha_0) = c(\alpha_2) \neq c(\alpha_1)$. A similar argument leads to $c(\alpha_4) = c(\alpha_6) \neq c(\alpha_5)$. It follows that $c(\alpha_0) = c(\alpha_2) = c(\alpha_5) \neq c(\alpha_1) = c(\alpha_4) = c(\alpha_6)$. Furthermore, if $\{\alpha_0, \alpha_1, \alpha_2\}$ contains s_a, t_a for some a, then G is type-B-partitionable; if $\{\alpha_1, \alpha_2, \alpha_4\}$ contains s_a, t_a for some a, then G is also type-B-partitionable, and so on. Thus, we can assume w.l.o.g. that $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$. A paired 3-DPC for the remaining case can be constructed as follows (see Fig. 7(c)):

1: For a vertex $x \in R_1$ with $c(x) = c(s_1)$, there exists a vertex $y \in R_0$ that admits a disjoint path cover composed of s_1–x and s_2–y paths in $R_{0,1}$.
2: For the neighbor $s_1' \in R_2$ of x, there exists a vertex $z \in R_4$ that admits a disjoint path cover composed of s_1'–t_1 and s_3–z paths in $R_{2,4}$.
3: For the neighbor $s_3' \in R_5$ of z and the neighbor $s_2' \in R_7$ of y, there exists a paired 2-DPC composed of s_2'–t_2 and s_3'–t_3 paths in $R_{5,7}$.
4: Concatenating the s_1–x and s_1'–t_1 paths results in an s_1–t_1 path; concatenating the s_2–y and s_2'–t_2 paths leads to an s_2–t_2 path; finally concatenating the s_3–z and s_3'–t_3 paths leads to an s_3–t_3 path.

The vertices y in Step 1 and z in Step 2 exist due to Theorem 4. The paired 2-DPC in Step 3 exists by Lemmas 4 and 6 (also, by Remark 1).

Finally, suppose $r \geq 4$. Then, G is row-reducible, or $m = 8$ and $r \in \{4, 5\}$. Let $m = 8$. If $r = 4$, then R_4 and R_7 contain no terminal, but each of R_5 and R_6 contains a single terminal, hence G is type-C-partitionable w.r.t. $R_{4,7}$. If $r = 5$, then R_6 contains a terminal but R_5 and R_7 does not. Let α_i denote the terminal in R_i again. If $c(\alpha_3) = c(\alpha_4)$, then G is type-C-partitionable w.r.t. $R_{3,5}$; also, if $c(\alpha_4) = c(\alpha_6)$, then G is type-C-partitionable w.r.t. $R_{4,7}$; in addition, if $c(\alpha_6) = c(\alpha_0)$, then G is type-C-partitionable w.r.t. $R_{5,7} \cup R_0$; finally, if $c(\alpha_0) = c(\alpha_1)$, then G is type-C-partitionable w.r.t. $R_7 \cup R_{0,1}$. It follows that $c(\alpha_3) \neq c(\alpha_4) \neq c(\alpha_6) \neq c(\alpha_0) \neq c(\alpha_1)$, and thus $c(\alpha_0) = c(\alpha_2) = c(\alpha_4) \neq c(\alpha_1) = c(\alpha_3) = c(\alpha_6)$. Furthermore, if $\{\alpha_0, \alpha_1, \alpha_2\}$ contains s_a, t_a for some a, then G is type-B-partitionable; if $\{\alpha_1, \alpha_2, \alpha_3\}$ contains s_a, t_a for some a, then G is also type-B-partitionable, and so on. Thus, we can assume w.l.o.g. that $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_3$, $t_2 \in R_4$, and $t_3 \in R_6$. The construction, shown below, is almost the same as in the previous case where $r = 3$, $m = 8$, $s_1 \in R_0$, $s_2 \in R_1$, $s_3 \in R_2$, $t_1 \in R_4$, $t_2 \in R_5$, and $t_3 \in R_6$.

1: For a vertex $x \in R_1$ with $c(x) = c(s_1)$, there exists a vertex $y \in R_0$ that admits a disjoint path cover composed of s_1–x and s_2–y paths in $R_{0,1}$
2: For the neighbor $s_1' \in R_2$ of x, there exists a vertex $z \in R_3$ that admits a disjoint path cover composed of s_1'–t_1 and s_3–z paths in $R_{2,3}$.
3: For the neighbor $s_3' \in R_4$ of z and the neighbor $s_2' \in R_7$ of y, there exists a paired 2-DPC composed of s_2'–t_2 and s_3'–t_3 paths in $R_{4,7}$.
4: Concatenating the s_1–x and s_1'–t_1 paths results in an s_1–t_1 path; concatenating the s_2–y and s_2'–t_2 paths leads to an s_2–t_2 path; finally, concatenating the s_3–z and s_3'–t_3 paths leads to an s_3–t_3 path.

This completes the entire proof. □

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REFERENCES

1. K. Asdre and S. D. Nikolopoulos, "The 1-fixed-endpoint path cover problem is polynomial on interval graphs," *Algorithmica*, vol. 58, no. 3, pp. 679-710, 2010.
2. S. C. Ntafos and S. L. Hakimi, "On path cover problems in digraphs and applications to program testing," *IEEE Transactions on Software Engineering*, vol. 5, no. pp. 520-529, 1979.
3. J. H. Park, H. C. Kim, and H. S. Lim, "Many-to-many disjoint path covers in hypercube-like interconnection networks with faulty elements," *IEEE Transactions on Parallel and Distributed Systems*, vol. 17, no. 3, pp. 227-240, 2006.
4. S. R. Arikati and C. P. Rangan, "Linear algorithm for optimal path cover problem on interval graphs," *Information Processing Letters*, vol. 35, no. 3, pp. 149-153, 1990.
5. J. H. Park, J. Choi, and H. S. Lim, "Algorithms for finding disjoint path covers in unit interval graphs," *Discrete Applied Mathematics*, vol. 205, pp. 132-149, 2016.
6. T. Dvorak and P. Gregor, "Partitions of faulty hypercubes into paths with prescribed endvertices," *SIAM Journal on Discrete Mathematics*, vol. 22, no. 4, pp. 1448-1461, 2008.
7. T. Dvorak, P. Gregor, and V. Koubek, "Generalized Gray codes with prescribed ends," *Theoretical Computer Science*, vol. 668, pp. 70-94, 2017.
8. S. Jo, J. H. Park, and K. Y. Chwa, "Paired many-to-many disjoint path covers in faulty hypercubes," *Theoretical Computer Science*, vol. 513, pp. 1-24, 2013.
9. X. B. Chen, "Paired 2-disjoint path covers of faulty k-ary n-cubes," *Theoretical Computer Science*, vol. 609, pp. 494-499, 2016.
10. X. B. Chen, "Paired 2-disjoint path covers of multi-dimensional torus networks with faulty edges," *Information Processing Letters*, vol. 116, no. 2, pp. 107-110, 2016.
11. B. G. Kronenthal and W. H. T. Wong, "Paired many-to-many disjoint path covers of hypertori," *Discrete Applied Mathematics*, vol. 218, pp. 14-20, 2017.
12. J. Li, G. Wang, and L. Chen, "Paired 2-disjoint path covers of multi-dimensional torus networks with 2n−3 faulty edges," *Theoretical Computer Science*, vol. 677, pp. 1-11, 2017.
13. H. S. Lim, H. C. Kim, and J. H. Park, "Ore-type degree conditions for disjoint path covers in simple graphs," *Discrete Mathematics*, vol. 339, no. 2, pp. 770-779, 2016.
14. I. Ihm and J. H. Park, "A linear-time algorithm for finding a paired 2-disjoint path cover in the cube of a connected graph," *Discrete Applied Mathematics*, vol. 218, pp. 98-112, 2017.
15. J. H. Park and I. Ihm, "Disjoint path covers in cubes of connected graphs," *Discrete Mathematics*, vol. 325, pp. 65-73, 2014.
16. F. Luccio and C. Mugnia, "Hamiltonian paths on a rectangular chessboard," in *Proceedings of the 16th Annual Allerton Conference on Communication, Control and Computing*, Urbana, IL, 1978, pp. 161-173.
17. K. Makino, "2-disjoint path covers in mesh-torus," *Research Institute for Mathematical Sciences*, vol. 1489, pp. 181-187, 2006.
18. J. H. Park and I. Ihm, "Many-to-many two-disjoint path covers in cylindrical and toroidal grids," *Discrete Applied Mathematics*, vol. 185, pp. 168-191, 2015.
19. H. D. Kim and J. H. Park, "Many-to-many disjoint path covers in two-dimensional bipartite tori with a single fault," *Journal of KIISE: Computer Systems and Theory*, vol. 39, no. 5, pp. 333-342, 2012.
20. G. J. Simmons, "Almost all n-dimensional rectangular lattices are Hamilton-Laceable," Sandia Labs., Albuquerque, NM, Technical Report, 1977.
21. C. C. Chen and N. F. Quimpo, "On strongly Hamiltonian abelian group graphsm" in *Combinatorial Mathematics VIII*. Heidelberg: Springer, 1981, pp. 23-34.
22. C. H. Tsai, J. M. Tan, Y. C. Chuang, and L. H. Hsu, "Fault-free cycles and links in faulty recursive circulant graphs," in *Proceedings of ICS Workshop on Algorithms and Theory of Computation*, Chiayi, Taiwan, 2000, pp. 74-77.

Jung-Heum Park

He received the B.S. degree in Computer Science and Statistics from Seoul National University in 1985, and the M.S. and Ph.D. degrees in Computer Science from KAIST, Korea, in 1987 and 1992, respectively. He joined IERI, KAIST as a postdoctoral researcher in 1992. During 1993-1996, he was a senior member of research staff at the ETRI. In 1996, he joined the Department of Computer Science at the Catholic University of Korea as an assistant professor, and currently he is a professor in the School of Computer Science and Information Engineering. His research interests include design and analysis of algorithms, applied graph theory, and interconnection networks. He is a member of the ACM, the IEEE, and the IEEE Computer Society.