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Minimum-Width Parallelogram Annulus with Given Angles

Sang Won Bae*
Division of Artificial Intelligence and Computer Engineering, Kyonggi University, Suwon, Korea
swbae@kgu.ac.kr

Abstract
In this paper, we study a variant of the problem of computing a minimum-width parallelogram annulus that encloses a given set of n points in the plane. A parallelogram annulus is a closed region between a parallelogram and its inward offset. Specifically, we present the first algorithm that computes a minimum-width parallelogram annulus with inner angles fixed by the input that encloses n input points. The running time is $O(n^2 \log n)$. To the best of our knowledge, there exists no known algorithm in the literature for the stated problem, and our algorithm generalizes the existing $O(n^2 \log n)$-time algorithm for the rectangular annulus in arbitrary orientation in the same running time-bound.

Category: Algorithms and Complexity

Keywords: Algorithms design and analysis; Computational geometry; Parallelogram annulus; Arbitrary orientation

I. INTRODUCTION

In the curve-fitting problem, we are given a set P of input points and a target curve class $\mathcal{C}$ and are required to find an optimal transformation that applies to the best curve $C \in \mathcal{C}$ so that the transformed curve best fits P by minimizing a prescribed error function. One possible solution to the curve-fitting problem, when $\mathcal{C}$ is a subclass of closed convex curves, can be found by the minimum-width annulus problem.

An annulus informally depicts a ring-shaped region in the plane that is often described by two concentric circles. One can consider a generalization to any convex shape $\mathcal{C}$, such as squares, rectangles, and even convex polygons (Fig. 1). The minimum-width annulus problem requires finding an annulus of a certain shape $\mathcal{C}$ with minimum width that encloses a given set P of n points in the plane. Among other shapes, the case when $\mathcal{C}$ consists of circles have been studied for the first time and the most intensively with an application to the roundness problem, thereby resulting in early $O(n^2)$-time solution [1-3]. The first sub-quadratic $O(n^{\frac{8}{5}+\varepsilon})$-time algorithm was presented by Agarwal et al. [4]. The current best exact algorithm takes $O(n^{\frac{3}{2}+\varepsilon})$ time as reported by Agarwal and Sharir [5].

Abellanas et al. [6] considered minimum-width rectangular annuli that are axis-parallel and presented two algorithms taking $O(n)$ or $O(n \log n)$ time: one minimizes the width over rectangular annuli with arbitrary aspect ratio and the other does over rectangular annuli with a prescribed aspect ratio, respectively. Gluchshenko et al. [7] presented an $O(n \log n)$-time algorithm that computes a minimum-width axis-parallel square annulus, and proved a matching lower bound, while the second algorithm by Abellanas et al. [6] can do the same in the same time-bound. If one considers rectangular or square annuli in arbitrary orientation, the problem becomes more difficult. Mukherjee et al. [8] presented an $O(n^2 \log n)$-time algorithm that computes a minimum-width rectangular annulus in arbitrary orientation and arbitrary aspect ratio. The author studied the problem of finding a minimum-width square annulus over all the orientations and presented the first algorithm with running time $O(n^3 \log n)$ [9]. Afterward, it

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*Corresponding Author

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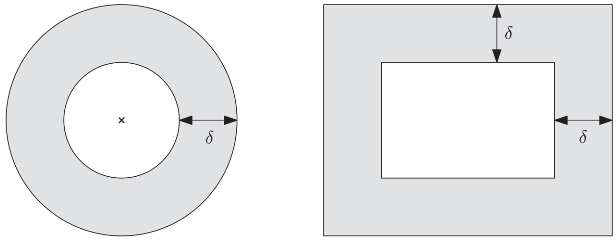

Fig. 1. A circular and a rectangular annulus of width δ.

was improved to $O(n^3)$ time by the author [10] and soon to $O(n^2 \log n)$ time by Bae and Yoon [11].

Recently, the parallelogram annulus was taken into account in the literature. A parallelogram annulus is defined to be the region between a parallelogram and its inward offset, and the parallelogram annulus problem requires finding a minimum-width parallelogram annulus that encloses a given set P of input points in the plane (Fig. 2). The author [12] presented the first algorithm for the problem over all side orientations, whose running time is $O(n^3 \log n)$. Note that a parallelogram has two independent side orientations and so does a parallelogram annulus. In the same paper, some restricted cases where one or two side orientations are fixed are shown to be solved faster in $O(n^2)$ and $O(n)$ time, respectively.

In this paper, we study another variant of the parallelogram annulus problem in which we want to compute a minimum-width parallelogram annulus with inner angles fixed by input α. A parallelogram has two distinct inner angles whose sum is equal to $180^\circ = \pi$. In this variant of the problem, we are given an additional input $0 < \alpha \leq \pi/2$ and asked to find an annulus defined by two parallelograms whose inner angles are α and $\pi - \alpha$, while each of the two side orientations can be chosen freely. We present an $O(n^2 \log n)$-time algorithm for this problem. To the best of our knowledge, there exists no known algorithm in the literature prior to this work. Note that when $\alpha = \pi/2$, our problem is exactly the same as the minimum-width rectangular annulus in arbitrary orientation. Hence, our result generalizes the $O(n^2 \log n)$-time algorithm by Mukherjee et al. [8] for the rectangular annulus problem in arbitrary orientation. Even better, our algorithm is simpler and easier to implement than that of Mukherjee et al. [8], as ours is based on basic algorithmic techniques without sophisticated data structures.

The rest of the paper is organized as follows: after introducing necessary preliminaries in Section II, we investigate parallelogram annuli with fixed inner angles and collect their properties in Section III. Our algorithm is described and analyzed in Section IV. Finally, Section V concludes the paper with some remarks and discussions.

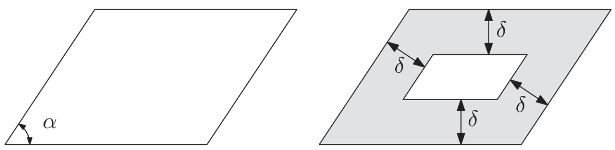

Fig. 2. A parallelogram and a parallelogram annulus of width δ with inner angles α and $\pi - \alpha$.

II. PRELIMINARIES

The orientation of a line or a line segment ℓ in the plane is a real number θ in the range $[0, \pi)$ such that the rotated copy of the x-axis by θ counter-clockwise is parallel to ℓ. If the orientation of a line or a line segment is θ, then we say that the line or line segment is *θ-aligned*. We regard the space $[0, \pi)$ of orientations homeomorphic to a circle, so any real number ϕ can be considered to be an orientation modulo π: that is, $\phi \equiv \theta$ for $\theta \in [0, \pi)$ if and only if $\phi = k\pi + \theta$ for some integer k.

A *parallelogram* is a quadrilateral with two pairs of parallel sides. For $\delta > 0$, an *(inward) offset* of a parallelogram R by δ is a smaller parallelogram $R' \subset R$ obtained by sliding each side of R inwards by distance δ. A *parallelogram annulus A* is the closed region between a parallelogram R and its offset R' by some $\delta > 0$, that is, $A = R \setminus \mathrm{int} R'$, where $\mathrm{int} R'$ denotes the interior of R'. Hence, the boundary of a parallelogram annulus A consists of two parallelograms R and R' with $R' \subset R$, called as the outer and the *inner parallelograms* of A, respectively. The *width* of A is defined to be δ, the distance between the outer and the inner parallelograms R and R'.

Note that a parallelogram has two distinct orientations of its sides and two distinct inner angles at its corners that sum up to π. For any parallelogram R, if the sides of R are θ-aligned and ϕ-aligned for some $0 \leq \theta < \phi < \pi$, then the inner angles at the corners of R are $\phi - \theta$ and $\pi - \phi + \theta$.

Throughout the paper, we demonstrate the parallelograms with fixed inner angles and annuli defined by these parallelograms. For an angle $0 < \alpha \leq \pi/2$, we call a parallelogram R an *α-parallelogram* if its inner angles are α and $\pi - \alpha$, and call a parallelogram annulus A an α-annulus if its outer and inner parallelograms are α-parallelograms. Also, an α-parallelogram is called θ-aligned if one of its sides is θ-aligned; an α-annulus is called θ-aligned if its outer and inner parallelograms are θ-aligned.

For any two points $p, q \in \mathbb{R}^2$, let $\overline{pq}$ denote the line segment joining p and q, and $|\overline{pq}|$ denote the length of $\overline{pq}$. We will discuss the distance between the two parallel lines through p and q, respectively, in some orientation $\theta \in [0, \pi)$. For any $p, q \in \mathbb{R}^2$ and $\theta \in [0, \pi)$, define $d_\theta(p, q)$ be the orthogonal distance between the two θ-aligned lines through p and q, respectively. From simple geometry, it is not difficult to see that

$$d_\theta(p, q) = |\overline{pq}| \cdot |\sin(\theta - \theta_{pq})|,$$

where $\theta_{pq} \in [0, \pi)$ denotes the orientation of $|\overline{pq}|$ (See Fig. 3).

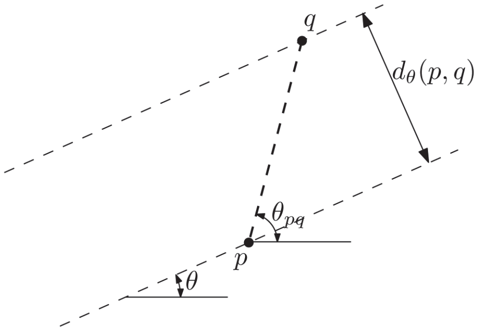

Fig. 3. Illustration to $d_\theta(p, q)$.

In this paper, we are interested in the following problem: given a set P of n points in the plane $\mathbb{R}^2$ and an angle $\alpha \in [0, \pi/2)$, compute an α-annulus that encloses P that minimizes its width over all the possible orientations $0 \in [0, \pi)$.

In the entire paper, we will often face the functions of a particular form $\alpha \sin(\theta + b)$ for some $a, b \in \mathbb{R}^2$. Such a function is called *sinusoidal functions*. Obviously, the equation $\alpha \sin(\theta + b) = 0$ has at most one zero over $\theta \in [0, \pi)$. The following property of sinusoidal functions is well known and easily derived.

OBSERVATION 1. The sum of two sinusoidal functions is also sinusoidal. Therefore, the graphs of two sinusoidal functions cross at the most once over domain $[0, \pi)$.

Observe that for a fixed pair of points $p, q \in \mathbb{R}^2$, the function $f(\theta) = d_\theta(p, q)$ is a piecewise sinusoidal function of $\theta \in [0, \pi)$ with at most one breakpoint at $\theta = \theta_{pq}$.

III. PROPERTIES OF PARALLELOGRAM ANNULI WITH FIXED INNER ANGLES

In this section, we prove some properties of parallelogram annuli that enclose a given set P of n points, which will be later used to devise our algorithm. In the following, let α be a given angle such that $0 < \alpha \leq \pi/2$. Note that we are interested only in α-annuli.

We first observe the following.

OBSERVATION 2. For each $\theta \in [0, \pi)$, there exists a minimum-width θ-aligned α-annulus A enclosing P such that every side of the outer parallelogram of A contains at least one point in P.

Proof. Consider any minimum-width θ-aligned α-annulus A enclosing P and let R be the outer parallelogram of A. Suppose that there is a side e of R that contains no point in P. Then, we can slide e and its corresponding side of the inner parallelogram simultaneously inwards, until e hits a point of P. Observe that the width of the annulus remains the same as that of A during this sliding process, and there is no point of P missing out as we have stopped as soon as e hits a point in P.

As a result, we now have a new annulus A' that encloses P and has the same width as A, while the new outer parallelogram contains one more point in P on its boundary. If there is another side of the outer parallelogram that contains no point of P, then we can repeat the above sliding process for that side. After executing the sliding process for about four times, we obtain another minimum-width θ-aligned α-annulus enclosing P such that every side of its outer parallelogram contains at least one point in P, as claimed. □

Note that the condition of Observation 2 does not mean that there are four distinct points in P on the boundary of the outer parallelogram; if there is a point $p \in P$ located at a corner, then both sides incident to the corner contain p. Consider a fixed orientation $\theta \in [0, \pi)$ and a minimum-width θ-aligned α-annulus A that encloses P. The sides of the outer parallelogram R of A are θ-aligned and $(\theta \pm \alpha)$-aligned, that is, if one side of R is θ-aligned, then the sides adjacent to it are either $(\theta + \alpha)$-aligned or $(\theta - \alpha)$-aligned.

Both cases are symmetric, so we assume the former case. The outer parallelogram R encloses P by definition. Let $R(\theta)$ be the smallest parallelogram that encloses P such that the sides of $R(\theta)$ are θ-aligned and $(\theta \pm \alpha)$-aligned.

OBSERVATION 3. For each $\theta \in [0, \pi)$, the smallest parallelogram $R(\theta)$ that encloses P is uniquely determined and every side of $R(\theta)$ contains at least one point of P.

Proof. Note that the side orientations of $R(\theta)$ are θ and $\theta + \alpha$. If a side of $R(\theta)$ contains no point of P, then we can perform the sliding process for the side as described in the proof of Observation 2 and this leads to a contradiction to the minimality of $R(\theta)$. Therefore, every side of $R(\theta)$ contains at least one point of P.

In order to see the uniqueness of $R(\theta)$, consider the smallest strip $S(\theta)$ that encloses P bounded by two θ-aligned lines that encloses P. Similarly, $S(\theta + \alpha)$ is the smallest strip that encloses P bounded by two $(\theta + \alpha)$-aligned lines. Then, $R(\theta) = S(\theta) \cap S(\theta + \alpha)$. Since $S(\theta)$ and $S(\theta + \alpha)$ are determined uniquely, we conclude that $R(\theta)$ is unique. □

Observations 2 and 3 imply that there is a minimum-width θ-aligned α-annulus enclosing P such that its outer parallelogram is either $R(\theta)$ or $R(\theta - \alpha)$. This also means that we can find the optimal θ-aligned α-annulus by trying only two candidate outer parallelograms. Once any outer parallelogram is chosen, the possible inner parallelogram for our purpose is also uniquely determined as described in the following section.

Suppose that $R(\theta)$ is chosen as the outer parallelogram for any $\theta \in [0, \pi)$. For each point $p \in P$, we define $\delta_p(\theta)$ to be the nonnegative real number such that p lies on the boundary of the offset of $R(\theta)$ by $\delta_p(\theta)$. We then define

$$w(\theta) := \max_{p \in P} \delta_p(\theta).$$

It can be seen that $\delta_p(\theta)$ is the minimum possible width of any parallelogram annulus containing p whose outer parallelogram is $R(\theta)$. Hence, for any such annulus in order to enclose all points in P, its width should be at least $\delta_p(\theta)$ for any $p \in P$, so $w(\theta) = \max \delta_p(\theta)$. This implies that the minimum-width annulus enclosing P whose outer parallelogram is $R(\theta)$ should have the width $w(\theta)$ exactly, and the corresponding inner parallelogram is also uniquely determined by the offset of $R(\theta)$ by $w(\theta)$. Let $R'(\theta)$ be the inner parallelogram and $A(\theta)$ be the corresponding annulus. Note that $R'(\theta) \subseteq R(\theta)$ is the offset of $R(\theta)$ by the width $w(\theta)$ of $A(\theta)$, and all points in P are enclosed in the θ-aligned α-annulus $A(\theta)$.

Lemma 1. *For any $\theta \in [0, \pi)$, the minimum-width parallelogram annulus enclosing P whose outer parallelogram is $R(\theta)$ is uniquely determined as $A(\theta)$ and its width is exactly $w(\theta)$. Either $A(\theta)$ or $A(\theta - \alpha)$ is a minimum-width θ-aligned α-annulus that encloses P that is described in Observation 2.*

Proof. This lemma immediately follows from the above discussion together with Observations 2 and 3. □

Now, we turn back to our original problem that requires finding a minimum-width α-annulus over all orientations. By Lemma 1, our problem is to find an optimal orientation $\theta^* \in [0, \pi)$ that minimizes $w(\theta^*)$. We observe that in such an optimal orientation θ^*, we have an additional point on the boundary of the annulus $A(\theta^*)$ as follows. See Fig. 4 for an illustration.

Lemma 2. *Let $\theta^* \in [0, \pi)$ be such that $w(\theta^*) = min_{\theta \in [0, \pi)} w(\theta)$. Then, one of the following properties holds:*

(a) *Two distinct points in P lie on a common side of the outer parallelogram $R(\theta^*)$.*
(b) *Two distinct points in P lie on the boundary of the inner parallelogram $R'(\theta^*)$, or a point in P lies at a corner of $R'(\theta^*)$.*

Proof. Consider the annulus $A(\theta^*)$ that is θ^*-aligned with outer and inner parallelograms $R(\theta^*)$ and $R'(\theta^*)$. Recall that each side of $R(\theta^*)$ contains a point in P by Observation 3 and that there is at least one point on the boundary of $R'(\theta^*)$ by definition.

Suppose to the contrary that no side of $R(\theta^*)$ contains two points in P and there is only one point $q \in P$ on the relative interior of the side of $R'(\theta^*)$. Without loss of generality, assume that the side of $R'(\theta^*)$ that contains q is θ-aligned. Let $p \in P$ be the point on the side of $R(\theta^*)$ such that $d_{\theta^*}(p, q) = w(\theta^*)$. Since $w(\theta^*) = d_{\theta^*}(p, q) = |\overline{pq}| \cdot |\sin(\theta - \theta_{pq})|$ and q is the only point on the boundary of $R'(\theta^*)$, there should exist a sufficiently small positive $\varepsilon > 0$ such that one of the following holds: $w(\theta^* + \varepsilon) = d_{\theta^*+\varepsilon}(p, q) < d_{\theta^*}(p, q) = w(\theta^*)$ or $w(\theta^* - \varepsilon) = d_{\theta^*-\varepsilon}(p, q) < d_{\theta^*}(p, q) = w(\theta^*)$. In either case, it leads to a contradiction to the assumption that θ^* minimizes the function w. Hence, the lemma follows. □

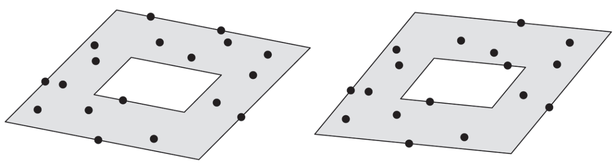

Fig. 4. (a) Two distinct points on a common side of the outer parallelogram. (b) Two distinct points on the boundary of the inner parallelogram.

IV. ALGORITHM

In this section, we describe our algorithm that exactly computes a minimum-width α-annulus enclosing P over all the possible orientations. Once we know an optimal orientation θ^*, defined as in the previous section, we can compute the corresponding α-annulus $A(\theta^*)$ in linear time using the parallelogram annulus algorithm for two fixed side orientations presented in [12]. Thus, our problem is reduced to the optimization problem that minimizes the function $w(\theta^*)$ over $\theta \in [0, \pi)$ and find an optimal orientation θ^*. For this purpose, we first analyze the function w in domain $[0, \pi)$ based on the properties revealed in the previous sections and then describe our algorithm.

Recall that $w(\theta) = \max_{p \in P} \delta_p(\theta)$, that is, the upper envelope of the functions δ_p over $[0, \pi)$. Also, note that $\delta_p(\theta)$ can be interpreted as the distance between p and its closest point to the boundary of the outer parallelogram $R(\theta)$. Since $R(\theta)$ has four sides, $\delta_p(\theta)$ is determined by the minimum of the distance from p to each of the four sides. Note that each side of $R(\theta)$ contains a point in P by Observation 3. Therefore, $\delta_p(\theta)$ is determined by and dependent on these points in P lying on the sides of $R(\theta)$. These points are known as *extreme* points of P.

A point in P is called *extreme* if it appears as a corner of the convex hull conv(P) of P. As in the proof of Observation 3, for each $\theta \in [0, \pi)$, define $S(\theta)$ to be the smallest strip that encloses P bounded by two parallel θ-aligned lines. Observe that conv(P)$\subset S(\theta)$ and that the two lines defining $S(\theta)$ are tangent to conv(P). Let $\ell_l(\theta)$ and $\ell_r(\theta)$ be the two lines bounding $S(\theta)$ such that $\ell_l(\theta)$ is to the left of $\ell_r(\theta)$ if $0 < \theta < \pi$; $\ell_l(0)$ is above $\ell_r(0)$ for $\theta = 0$. Define $\chi_l(\theta)$ and $\chi_r(\theta)$ to be the points in P lying on $\ell_l(\theta)$ and $\ell_r(\theta)$, respectively. Observe that $\ell_l(\theta)$ and $\ell_r(\theta)$ are extreme points of P. See Fig. 5.

Since $R(\theta) = S(\theta) \cap S(\theta + \alpha)$ as shown in the proof of Observation 3, their corresponding extreme points $\chi_l(\theta)$, $\chi_r(\theta)$, $\chi_l(\theta+\alpha)$, and $\chi_r(\theta+\alpha)$ are located on the sides of $R(\theta)$. Note that these four extreme points may not be distinct. From the above discussion, this implies that

$$\delta_p(\theta) = \min\{d_\theta(p, \chi_l(\theta)), d_\theta(p, \chi_r(\theta)), d_{\theta+\alpha}(p, \chi_l(\theta+\alpha)), d_{\theta+\alpha}(p, \chi_r(\theta+\alpha))\}$$

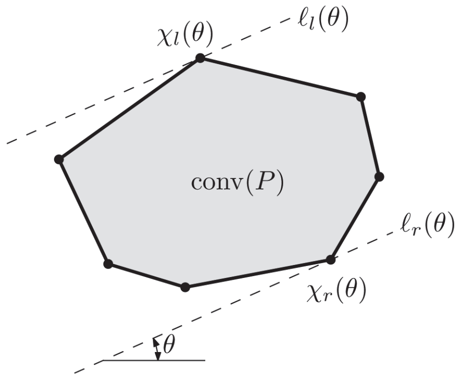

Fig. 5. Extreme points $\chi_l(\theta)$ and $\chi_r(\theta)$ lying on the θ-aligned lines $\ell_l(\theta)$ and $\ell_r(\theta)$ tangent to conv(P).

for any $p \in P$ and $\theta \in [0, \pi)$.

On the other hand, $\chi_l(\theta)$ and $\chi_r(\theta)$ as functions of $\theta \in [0, \pi)$ change at most n times. This can be easily shown as follows: consider the line $\ell_l(\theta)$ at $\theta = 0$ and its motion as θ increases continuously to π. We observe that $\ell_l(\theta)$ turns around the convex hull conv(P) and the extreme point $\chi_l(\theta)$ lying on ℓ_l only changes exactly when $\chi_l(\theta)$ contains an edge of conv(P). This means that the orientation space $[0, \pi)$ can be decomposed into $O(n)$ intervals I such that the tuple ($\chi_l(\theta)$, $\chi_r(\theta)$, $\chi_l(\theta+\alpha)$, $\chi_r(\theta+\alpha)$) is fixed over all $\theta \in I$. We call such an interval $I \subset [0, \pi)$ a *primary interval*. We then observe the following.

Lemma 3. *Let $I \subset [0, \pi)$ be any primary interval. Then, for any $p \in P$, the function $\delta_p(\theta)$ over the domain $\theta \in I$ is piecewise sinusoidal with at most three breakpoints.*

Proof. By the definition of primary intervals, the tuple ($\chi_l(\theta)$, $\chi_r(\theta)$, $\chi_l(\theta+\alpha)$, $\chi_r(\theta+\alpha)$) is fixed over all $\theta \in I$. Let $\chi_l := \chi_l(\theta)$, $\chi_r := \chi_r(\theta)$, $\chi_l' := \chi_l(\theta+\alpha)$, $\chi_r' := \chi_r(\theta+\alpha)$ for $\theta \in I$. Then from the discussions above, we have

$$\delta_p(\theta) = \min\{d_\theta(p, \chi_l), d_\theta(p, \chi_r), d_{\theta+\alpha}(p, \chi_l'), d_{\theta+\alpha}(p, \chi_r')\}$$

for any $p \in P$ and $\theta \in I$.

By definition, each of the four functions in the "min" of the above equation is a sinusoidal function of θ. That is, δ_p is the lower envelope of the four sinusoidal functions over domain I. By Observation 1, the graphs of any two of them cross at most once. This implies that each of them can appear at most once in their lower envelope δ_p. See Fig. 6 for an illustration. Hence, the lemma follows. □

Now, we are ready to describe our algorithm. Our algorithm runs in two phases: in the first phase, compute the primary intervals and the description of functions δ_p for all $p \in P$ in every primary interval I and then, in the second phase, we compute their upper envelope $w(\theta)$ over $\theta \in [0, \pi)$ and find the minimum of $w(\theta)$ to find an optimal orientation θ^*. More details are given in the following section.

In the first phase, we start with computing the convex hull conv(P) of P and then we identify the primary intervals. This can be done simply by turning around the boundary of conv(P) and listing the orientations of the edges of conv(P) in order. During this process, we also identify the four extreme points ($\chi_l(\theta)$, $\chi_r(\theta)$, $\chi_l(\theta+\alpha)$, $\chi_r(\theta+\alpha)$) for each primary interval I and any $\theta \in I$. Next, for each primary interval I and all $p \in P$, compute the description of function $\delta_p(\theta)$ over $\theta \in I$. By Lemma 3, the graph of $\delta_p(\theta)$ consists of at most four sinusoidal curves over $\theta \in I$. What remains in the first phase is collecting all these sinusoidal curves for all $p \in P$ and all primary intervals I. Let Γ be the set of those collected curves.

Lemma 4. *The set Γ consists of $O(n^2)$ sinusoidal curves. The first phase of our algorithm takes $O(n^2)$ to compute Γ.*

Proof. Computing the convex hull of n points takes $O(n \log n)$ time [13]. Identifying the primary intervals and the extreme points can be done by walking around the boundary of conv(P) and this takes only $O(n)$ time. For each primary interval I and $p \in P$, computing the description of function δ_p takes $O(1)$ time and the function δ_p consist of at most four sinusoidal curves by Lemma 3. Since there are $O(n)$ primary intervals and P consists of n points, this takes $O(n^2)$ time in total to collect all those $O(n^2)$ sinusoidal curves. Hence, the lemma follows. □

In the second phase of our algorithm, we compute the upper envelope of Γ, resulting in the full description of function $w(\theta)$ over $\theta \in [0, \pi)$. Since our goal is to minimize $w(\theta)$, we are done by searching the description of $w(\theta)$.

Lemma 5. *The upper envelope of Γ consists of $O(n^2\alpha(n))$ sinusoidal curves and it can be computed in $O(n^2 \log n)$ time, where $\alpha(\cdot)$ denotes the inverse Ackermann function.*

Proof. Any curve in Γ is the graph of a partial sinusoidal function. Thus, any two curves in Γ cross at most once by Observation 1. The upper envelope of such curves can be computed in $O(n^2 \log n)$ time by the algorithm of Hershberger [14], while its complexity is known to be at most $O(n^2\alpha(n))$ from the theory of Davenport–Schinzel sequences [15]. □

The last task is just searching for the lowest point of the upper envelope computed by Lemma 5, which corresponds to the optimal orientation θ^* that minimizes $w(\theta)$. By Lemma 2, we know that such a lowest point of the upper envelope occurs only at a breakpoint of w, that is, an intersection point of two sinusoidal curves in Γ. Therefore, the optimal orientation θ^* can be found by checking all the vertices in the upper envelope in order.

We finally conclude the main theorem.

Theorem 1. *Given a set P of n points and an angle $\alpha \in (0, \pi/2]$, a minimum-width parallelogram annulus* *enclosing P whose inner angles are α and π − α can be computed in* $O(n^2 \log n)$ *time.*

Proof. The correctness of our algorithm is proved by the above discussions. In order to see the time complexity, we take $O(n^2)$ time for the first phase. In the second phase, we spend $O(n^2 \log n)$ time to compute the upper envelope of sinusoidal curves in Γ as shown in Lemma 5. And then we scan all the vertices of the upper envelope to find an optimal orientation θ^*. This can be done in time linear to the complexity of the upper envelope, which is $O(n^2\alpha(n))$ by Lemma 5. Since $\alpha(n) = O(\log n)$, the total time complexity of our algorithm is bounded by $O(n^2 \log n)$, as claimed. □

V. CONCLUDING REMARKS

We have presented the first algorithm that computes a minimum-width parallelogram annulus with a given inner angle α that encloses a given set of points. Our algorithm takes $O(n^2 \log n)$ time, and this matches the time complexity of the best-known algorithm that computes a minimum-width rectangular annulus over all the orientations [8]. Note that rectangular annuli are a special form of parallelogram annuli with a given inner angle $\alpha = \pi/2$. Hence, we have generalized the previous result while keeping the running time.

Our algorithm is efficient yet easy to implement. The most involved step is the second phase, in which the upper envelope of sinusoidal curves Γ is computed by invoking the algorithm of Hershberger [14]. Note that Hershberger's algorithm runs in a divide-and-conquer fashion with a basic tree structure.

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Sang Won Bae

Sang Won Bae got his PhD in 2008 at Department of Computer Science, Korea Advanced Institute of Science and Technology, Daejeon, Korea. At present, he is working as a professor at Division of Artificial Intelligence and Computer Engineering, Kyonggi University, Suwon, Korea. Research interests include algorithms design and analysis in computational geometry, discrete and combinatorial geometry, graph theory, and their algorithmic applications to other disciplines.