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On Counting Monotone Polygons and Holes in a Point Set

Sang Won Bae*
Division of Artificial Intelligence and Computer Science, Kyonggi University, Suwon, Korea
swbae@kgu.ac.kr

Abstract
In this paper, we study the problem of counting the number of monotone polygons in a given set S of n points in general position in the plane. A simple polygon is said to be monotone when any vertical line intersects its boundary at most twice. To our best knowledge, this counting problem remains unsolved and no nontrivial algorithm is known so far. As a research step forward to tackle the problem, we define a subclass of monotone polygons and present the first efficient algorithms that exactly count them.

Category: Computer Graphics / Image Processing

Keywords: Erdös–Szekeres problem; Counting problem; Monotone polygon; Discrete geometry; Computational geometry; Algorithm

I. INTRODUCTION

Let S be a set of n points in the plane in general position such that no three points in S are collinear. We are interested in simple polygons whose corners are chosen from points in S. Such a polygon with k corners is called a *k-gon* in the point set S, or we say that S *contains* it. If a k-gon in S contains no other point in S in its interior, then we say it is *empty*. A *k-hole* is an empty k-gon.

Combinatorial and algorithmic studies on k-gons and k-holes were initiated by Erdös and Szekeres [1] in 1935, and are still conducted by many researchers today. In particular, k-gons and k-holes enumeration and counting problems have been intensively studied in computational geometry. Note that counting 3-gons (triangles) in S is trivial as every triple in S forms a triangle. Rote et al. [2] presented an $O(n^{k-2})$-time algorithm that exactly counts the number of convex k-gons for any $k \geq 4$, and Rote and Woeginger [3] improved it to $O(n^{\lceil k/2 \rceil})$ time. Later, Mitchell et al. [4] came up with a dynamic programming approach, resulting in an $O(kn^3)$-time algorithm that counts the number of convex k-gons for any given $k \geq 4$. Note that counting convex 4-gons in S can be done in $O(n^2)$ time by earlier results [2, 3].

Mitchell et al. [4] also considered the problem of counting convex holes. They presented two algorithms that compute the number of all convex holes in S and the number of convex k-holes for a given integer $k \geq 3$ in $O(n^3)$ and $O(kn^3)$ time, respectively. Their second algorithm indeed computes the number of convex l-holes for every $3 \leq l \leq k$ in the same time bound. Earlier, Dobkin et al. [5] presented an algorithm that enumerates all convex k-holes in time $O(T+k \cdot H)$, where T denotes the number of empty triangles in S and H denotes the number of convex k-holes in S. Note that their algorithm, in particular for $k=3$, takes $O(T)$ time. While the number T of empty triangles in S can be as large as $\Theta(n^3)$, its expected value is known to be $\Theta(n^2)$ when the n points in S are chosen uniformly and independently at random from a convex and bounded body [6, 7]. This

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*Corresponding Author

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implies that Dobkin et al.'s enumeration algorithm practically runs faster than Mitchell et al.'s algorithm [4] to count the number T of empty triangles (3-holes). Recently, the author [8] improved the above algorithms by combining the approaches of Dobkin et al. [5] and Mitchell et al. [4] such that counting all convex holes can be done in $O(T)$ time and counting all convex k-holes can be done in $O(k \cdot T)$ time.

Compared to these fruitful results of counting convex polygons and holes in a given point set, it seems quite hard to tackle the problem of counting other popular classes of polygons and holes including *non-convex* ones. It was only recently that Sung and Bae [9, 10] succeeded in devising efficient algorithms that count the number of *non-convex* 4-holes and 5-holes. Note that these are the first known algorithms that count non-convex holes in a given planar set S, and it is still unknown how to efficiently count the number of non-convex k-holes for $k \geq 6$. Furthermore, to our best knowledge, we are unaware of any nontrivial algorithm that counts the number of all non-convex polygons or all non-convex holes.

A. Our Results and Contribution

As the next research step towards counting non-convex polygons and holes, we initiate a study of counting *monotone* polygons and holes in a given set S. A polygon is said to be monotone if any vertical line in the plane intersects it in at most one connected component. Note that there is no known algorithm that counts the number of monotone polygons and holes in S. In this paper, we defined a proper subclass of monotone polygons, called *LT-monotone* polygons that admit a greedy triangulation by recursively taking the leftmost triangle. For the first time, we present efficient algorithms that count the number of LT-monotone polygons and holes in a given set S of n points in general position. Our results are summarized as follows:

1. The number of all LT-monotone polygons in S can be counted in $O(n^3)$ time.
2. For $k \geq 3$, the number of LT-monotone k-gons in S can be counted in $O(kn^3)$ time.
3. The number of all LT-monotone holes in S can be counted in $O(T)$ time.
4. For $k \geq 3$, the number of LT-monotone k-holes in S can be counted in $O(kT)$ time.

The space complexity of our algorithms is all bounded by $O(n^2)$. To our best knowledge, there are no known algorithms for these counting problems on monotone polygons, and these are the first nontrivial algorithms that count a subclass of monotone polygons including all convex ones. Thus, our results are important progress as a solution to tackle those counting problems for a more general class of polygons. Note that the number T of empty triangles (i.e., 3-holes) varies from $\Omega(n^2)$ to $O(n^3)$ depending on the set S, and it is known that the expected value of T is $\Theta(n^2)$ when the points in S are chosen uniformly and independently at random from any convex and compact body [6, 7]. Hence, our algorithms counting monotone holes run in $O(n^2)$ and $O(kn^2)$ expected time, respectively, if S is given. as a random set.

B. Related Work

In 1935, Erdös and Szekeres [1, 11] asked the least number $N(k)$ for $k \geq 3$ such that any set of $N(k)$ points in general position in $\mathbb{R}^2$ contains a *convex* k-gon. This problem is known as the *Erdös–Szekeres problem*. It is proven that $N(4) = 5$, $N(5) = 9$ [1], and $N(6) = 17$ [12]. The exact value of $N(k)$ for $k \geq 7$ is not known. Erdös and Szekeres [1] also proved an upper bound $N(k) \leq \binom{2k-4}{k-2} + 1$ conjectured that $N(k) = 2^{k-2} + 1$ for each $k \geq 2$.

Note that the conjecture holds for $k \leq 6$. Erdös and Szekeres [11] later proved a matching lower bound $N(k) \geq 2^{k-2} + 1$ in 1961. Suk [13] in 2017 made a breakthrough $N(k) \leq 2^{k+O(k^{2/3}\log k)}$. Currently, the best upper bound is $N(k) \leq 2^{k+O(\sqrt{k\log k})}$ due to Holmsen et al. [14].

In 1978, Erdös also posed an empty version of the Erdös–Szekeres problem which asks the least number $H(k)$ for $k \geq 3$ such that any set of $H(k)$ points in general position contains a convex k-hole [15]. Some known results on $H(k)$ are as follows: $H(3) = 3$ and $H(4) = 5$ are easy to see, $H(5) = 10$ is proved by Harborth [16], $H(6)$ is bounded [17, 18], and Horton [19] proved that $H(k)$ for $k \geq 7$ is unbounded and undefined.

As a variant of the Erdös–Szekeres problem, the possible number of convex k-gons and k-holes in a set of n points has been extensively studied in the literature. It is not difficult to see that the maximum possible number of convex k-holes and convex k-gons is $\binom{n}{k} = \Theta(n^k)$ when the points in S are in convex position. Thus, most of the research studies have focused on the minimum possible numbers. Some important bounds are as follows. Barany and Furedi [6] showed that the minimum number of k-holes over all sets S of n points is $\Theta(n^2)$ for $k \in \{3, 4\}$ and is between $\Omega(n)$ and $O(n^2)$ for $k \in \{5, 6\}$. Note that for any $k \geq 7$, there exist some sets S of any size n such that there is no k-hole in S by Horton [19]. There has been constant efforts to improve the hidden constants in the $O(\cdot)$ and $\Omega(\cdot)$. For more history, see Barany and Valtr [20], Valtr [21], Dumistrescu [22], Pinchasi et al. [23], and references therein. A recent breakthrough was due to Aichholzer et al. [24] who showed that the minimum possible number of convex 5-holes is at least $\Omega(n\log^{4/5}n)$. Barany and Furedi [6] showed that the expected number T of empty triangles is $\Theta(n^2)$ when the points in S are chosen uniformly from a unit square. Recently, Balko et al. [7] extended these results to any higher-dimensional space and any convex body of unit measure from which points are chosen uniformly at random.

Despite a huge amount of results on convex k-gons and k-holes, it was relatively recent that the first nontrivial bounds on the minimum and maximum numbers of *non-convex* k-gons and k-holes have been proved. Aichholzer et al. [25, 26] established several new bounds on the maximum/minimum possible number of non-convex k-gons and k-holes for $k \geq 4$. Fabila-Monroy et al. [27] proved a sharp bound on the expected number of non-convex 4-gons.

II. PRELIMINARIES

Throughout the paper, we considered a standard coordinate system in the plane $\mathbb{R}^2$ with the x- and the y-axes. For any two points, $a, b \in \mathbb{R}^2$, we say that a is to the *left* of b or b is to the *right* of a if the x-coordinate of a is smaller than that of b. We denote by $\overline{ab}$, the line segment joining two points a and b. For any three distinct points, $p, q, r \in \mathbb{R}^2$, we say that (p, q, r) *makes a left turn* if r lies on the left side of the directed line through p towards q; or say that (p, q, r) *makes a right turn* if r lies on the right side of the directed line.

Let S be a set of n points in the plane $\mathbb{R}^2$ in general position, that is, no three points in S are collinear. Consider any simple polygon P in S, that is, a simple polygon whose corners form a subset of S. We say that P is *monotone* if any vertical line intersects P in at most one connected component, that is, either an empty set, a point, or a line segment. The *leftmost corner* of P, denoted by $l(P)$, is the corner of P with the lexicographically minimum coordinates. Hence, $l(P)$ is the corner with the minimum x-coordinate, while if there are two corners of P with the minimum x-coordinate, then $l(P)$ chooses the one with the smaller y-coordinate by definition. The *rightmost corner* of P is also defined in the analogous way to be the corner with the lexicographically maximum coordinates. The boundary of any monotone polygon P can be decomposed into two polygonal chains, called the *upper* and the *lower* chains, by cutting P at its leftmost and the rightmost corners.

Consider any monotone polygon P. Let q^+ and q^- be the corners of P adjacent to $l(P)$ on the upper chain and the lower chain of P, respectively. The triangle Δq^+pq^- is called the *leftmost triangle* of P. A monotone polygon P is called *LT-monotone* if it can be triangulated by recursively cutting out the leftmost triangle. More precisely, LT-monotone polygons are defined as follows: Let P be any monotone polygon. If P is a triangle, then P is LT-monotone. Otherwise, if the leftmost triangle Δ of P is completely contained in P and $P \backslash \Delta$ is LT-monotone, then P is also LT-monotone (see Fig. 1 for an illustration).

III. COUNTING LT-MONOTONE POLYGONS

We start by sorting S in the lexicographical order of the coordinates (x, y) of the points in S. Let $S = \{p_1, p_2, \ldots, p_n\}$ be in this order.

For any $k \geq 3$, let $g_k(S)$ be the number of LT-monotone k-gons in S, and $g(S)$ be the number of all LT-monotone polygons in S, that is, $g(S) = g_3(S) + g_4(S) + \ldots + g_n(S)$. In this section, we present how to efficiently compute $g(S)$ and $g_k(S)$ for a given $k \geq 3$.

A. Counting All LT-Monotone Polygons

For each $1 \leq a, b \leq n$, we defined the following quantities with an abuse of notation:

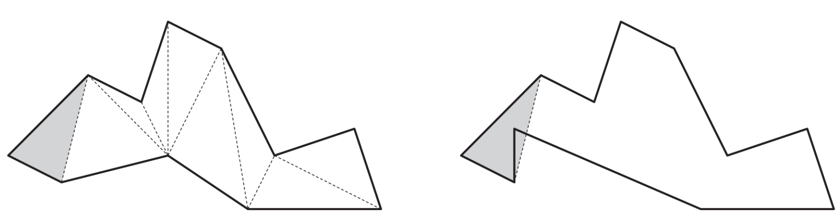
- Let $g(l; a, b)$ be the number of all LT-monotone polygons P in S such that the leftmost triangle of P is $\Delta p_l p_a p_b$ while p_a lies on the lower chain and p_b lies on the upper chain of P.
- Let $g^-(l; a)$ be the number of all LT-monotone polygons P in S such that $l(P) = p_l$ and p_a is the corner adjacent to p_l on the lower chain of P.
- Let $g^+(l; b)$ be the number of all LT-monotone polygons P in S such that $l(P) = p_l$ and p_b is the corner adjacent to p_l on the upper chain of P.

By definitions, note that

$$g^+(l; b) = \sum_{1 \leq a \leq n} g(l; a, b), \quad g^-(l; a) = \sum_{1 \leq b \leq n} g(l; a, b),$$

and

$$g(S) = \sum_{1 \leq l \leq n} \sum_{1 \leq i \leq n} g^+(l; i) = \sum_{1 \leq l \leq n} \sum_{1 \leq i \leq n} g^-(l; i).$$



(a) (b)

Fig. 1. An LT-monotone polygon (a) and a monotone polygon that is not LT-monotone (b). The shaded triangles of the two monotone polygons are the leftmost triangles.

We then establish a recurrence on $g(l; a, b)$ as follows:

Lemma 1. *For any $1 \leq l, a, b \leq n$, it holds that*

$$g(l; a, b) = \begin{cases} 0 & \text{if } l \geq \min\{a, b\} \text{ or } a = b \\ 0 & \text{if } (p_b, p_l, p_a) \text{ makes a right turn} \\ 1 + g^+(a; b) & \text{if } a < b \\ 1 + g^-(b; a) & \text{if } a > b \end{cases}$$

Proof. We start by checking the base cases. Suppose that $g(l; a, b) \geq 1$, so there exists an LT-monotone polygon P in S with $l(P) = p_l$ such that the two corners adjacent to p_l are p_a and p_b on the lower and the upper chains of P, respectively. This implies that $l < \min\{a, b\}$ and $a \neq b$ since P is a simple polygon. In addition, (p_b, p_l, p_a) make a left turn since p_l is the leftmost corner of P and p_a and p_b lie on the lower and the upper chains of P, respectively. This proves that if either $a = b$, $l \geq \min\{a, b\}$, or (p_b, p_l, p_a) make a right turn, then it holds that $g(l; a, b) = 0$.

Next, we consider the other cases, where $l < \min\{a, b\}$, $a \neq b$, and (p_b, p_l, p_a) makes a left turn. Let $\mathcal{G}$ be the set of all LT-monotone polygons P such that $l(P) = p_l$ and the two corners adjacent to p_l are p_a and p_b lying on the lower and the upper chains of P, respectively. Note that $|\mathcal{G}| = g(l; a, b)$. In this case, observe that $\mathcal{G} \neq \emptyset$ as the triangle $\Delta p_l p_a p_b \in \mathcal{G}(l; a, b)$, so $g(l; a, b) \geq 1$.

Here, we first assume that $a < b$, and prove that $g(l; a, b) = 1 + g^+(a; b)$. For this purpose, consider the set $\mathcal{G}'$ of all LT-monotone polygons P such that $l(P) = p_a$ and p_b is the corner adjacent to p_a on the upper chain of P. Note that $|\mathcal{G}'| = g^+(a; b)$. In the following, we establish a one-to-one correspondence between $\mathcal{G}'$ and $\mathcal{G} \backslash \{\Delta p_l p_a p_b\}$, concluding that $g(l; a, b) = 1 + g^+(a; b)$.

Consider any $P \in \mathcal{G}$ with $P \neq \Delta p_l p_a p_b$. Let P' be the polygon obtained by cutting out the leftmost triangle $\Delta p_l p_a p_b$ from P. Since P consists of at least four corners, P' is indeed an LT-monotone polygon in S with at least three corners, by the definition of LT-monotone polygons. In addition, we observe that $l(P') = p_a$ and the corner adjacent to p_a on the upper chain of P' is p_b (see Fig. 2(a)). This implies that $P' \in \mathcal{G}'$, and hence $g(l; a, b) - 1 \leq g^+(a; b)$.

We then consider any $Q' \in \mathcal{G}'$, and let $Q := Q' \cup \Delta p_l p_a p_b$. Since $l(Q') = p_a$ and $\overline{p_a p_b}$ is a side of Q', Q indeed forms a simple polygon with at least four corners. In addition, we observe that Q is LT-monotone by construction, $l(Q) = p_l$ as $l < a < b$, p_a is the corner adjacent to p_l on the lower chain of Q and p_b is the corner adjacent to p_l on the upper chain of Q. This implies that $Q \in \mathcal{G} \backslash \{\Delta p_l p_a p_b\}$, so $g(l; a, b) - 1 \geq g^+(a; b)$. This concludes that $g(l; a, b) = 1 + g^+(a; b)$, as claimed.

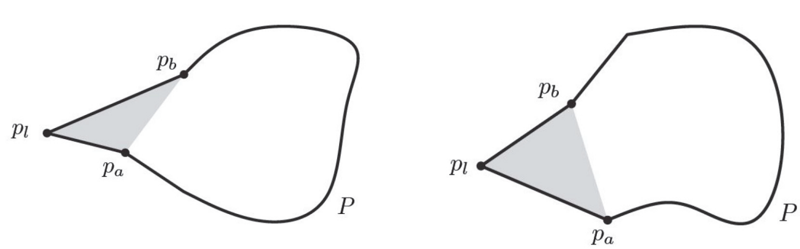
The other case where $a > b$ can be handled symmetrically and analogously by considering the set $\mathcal{G}'$ of all LT-monotone polygons P such that $l(P) = p_b$ and p_a is the corner adjacent to p_b on the lower chain of P, so that $|\mathcal{G}'| = g^-(b; a)$. As it was done above, we can establish a one-to-one correspondence between $\mathcal{G} \backslash \{\Delta p_l p_a p_b\}$ and $\mathcal{G}'$, concluding that $g(l; a, b) = 1 + g^-(b; a)$ if $a > b$ (see Fig. 2(b)). This completes the proof of the lemma. □

Now, we are ready to describe our algorithm to compute the number $g = g(S)$ of all LT-monotone polygons in S. Our algorithm is a dynamic programming algorithm based on the recurrence described in Lemma 1, by computing $g(l; a, b)$ for all $1 \leq l, a, b \leq n$ in the bottom-up fashion. We handle two two-dimensional $n \times n$ arrays G^+ and G^-, which will at last store the values of $g^+(i; j)$ and $g^-(i; j)$ in $G^+[i, j]$ and $G^-[i, j]$, respectively, for $1 \leq i, j \leq n$.

The description of our algorithm is as follows:

(1) Initialize G^+ and G^- with all entries set to zero.
(2) For each $l = n, n - 1, \ldots, 1$ in this order,
  (2–1) for each (a, b) such that $a \neq b > l$ and (p_b, p_l, p_a) makes a left turn,
    let $t \leftarrow 1 + G^+[a, b]$ if $a < b$
    or $t \leftarrow 1 + G^-[b, a]$ if $a > b$, and
    set $G^+[l, b] \leftarrow G^+[l, b] + t$ and
    $G^-[l, a] \leftarrow G^-[l, a] + t$.
(3) Compute the sum of all entries in G^+.
(4) Return it as the number $g(S)$ of all LT-monotone polygons in S.

Theorem 1. *Given a set S of n points in the plane in general position, the number $g(S)$ of all LT-monotone polygons in S can be computed in $O(n^3)$ time and $O(n^2)$ space.*



(a) (b)

Fig. 2. Proof of Lemma 1. (a) When $a < b$, the subpolygon $P' = P \backslash \Delta p_l p_a p_b$ is an LT-monotone polygon in S with $l(P') = p_a$. (b) When $a > b$, the subpolygon $P' = P \backslash \Delta p_l p_a p_b$ is an LT-monotone polygon in S with $l(P') = p_b$.

Proof. It is not difficult to see the time and space complexity of our algorithm described above are $O(n^3)$ and $O(n^2)$, respectively. Thus, we focus on showing the correctness of our algorithm. More specifically, we prove the following claim: for each value of $1 \le l \le n$, after executing step (2–1) of our algorithm, it holds that $G^+[l, i] = g^+(l; i)$ and $G^-[l, i] = g^-(l; i)$ for every $1 \le i \le n$.

It is trivially true when $l = n$, because $g^+(n; n) = g^-(n; n) = 0$ and $G^+[n, n] = G^-[n, n] = 0$ as initialized. Note in this case that there is no such pair (a, b) with $a, b > l$, so no updates in G^+ and G^- happened.

Now, assume that $l < n - 1$ and it holds that $g^+(l'; i) = G^+[l'; i]$ and $g^-(l'; i) = G^-[l', i]$ for every $l \le l'$, $i \le n$.

Then, in the loop for (a, b), our algorithm iterates all pairs (a, b) such that $a \neq b > l$ and (p_b, p_l, p_a) makes a left turn. For each such (a, b), it holds that $g(l; a, b) = 1 + g^+(a; b) = 1 + G^+[a, b]$ if $a < b$ or $g(l; a, b) = 1 + g^-(b; a) = 1 + G^-[b, a]$ if $a > b$ by Lemma 1 and the assumption. Hence, our algorithm correctly computes the value of $g(l; a, b)$ for every pair (a, b) and stores it in the variable t in step (2–1). As $g^+(l; b) = \sum_{1 \le a \le n} g(l; a, b)$ and $g^-(l; a) = \sum_{1 \le b \le n} g(l; a, b)$ after iterating all such pairs (a, b), our algorithm guarantees that $G^+[l, i]$ stores the exact value of $g^+(l; i)$ and $G^-[l, i]$ stores the exact value of $g^-(l; i)$ for every $l < i \neq n$. For each $i \le l$, it is obvious that $g^+(l; i) = g^-(l; i) = 0$ by Lemma 1, and the values of $G^+[l, i]$ and $G^-[l, i]$ also remain zero as initialized, since no updates on the values of these entries happen by the algorithm. Hence, the algorithm correctly computes all the values of $g^+(l; i)$ and $g^-(l; i)$ and stores them in $G^+[l, i]$ and $G^-[l, i]$ for all $1 \le l, i \le n$. □

B. Counting all LT-Monotone k-gons

As it was done above for $g(S)$, we define analogous terms for $g_k(S)$ as follows. For each $k \ge 3$ and each $1 \le l, a, b \le n$,

- Let $g_k(l; a, b)$ be the number of all LT-monotone k-gons P in S such that the leftmost triangle of P is $\Delta p_l p_a p_b$, while p_a lies on the lower chain and p_b lies on the upper chain of P,
- $g_k^-(l; a) := \sum_b g_k(l; a, b)$ be the number of all LT-monotone k-gons P in S such that $l(P) = p_l$ and p_a is the corner adjacent to p_l on the lower chain of P , and
- $g_k^+(l; b) := \sum_a g_k(l; a, b)$ be the number of all LT-monotone k-gons P in S such that $l(P) = p_l$ and p_b is the corner adjacent to p_l on the upper chain of P.

By definitions, note that

$$g_k^+(l; b) := \sum_{1 \le a \le n} g_k(l; a, b), \quad g_k^-(l; a) := \sum_{1 \le b \le n} g_k(l; a, b),$$

and

$$g_k(S) = \sum_{1 \le l \le n} \sum_{1 \le i \le n} g_k^+(l; i) = \sum_{1 \le l \le n} \sum_{1 \le i \le n} g_k^-(l; i).$$

We can establish a recurrence on $g_k(l; a, b)$ in a similar way as done in Lemma 1.

Lemma 2. *For any $k \ge 3$ and $1 \le l, a, b \le n$, it holds that*

$$g_k(l; a, b) = \begin{cases} 0 & \text{if } l \ge \min\{a, b\} \text{ or } a = b \\ 0 & \text{if } (p_b, p_l, p_a) \text{ makes a right turn} \\ 1 & \text{if } k = 3 \\ g_{k-1}^+(a; b) & \text{if } a < b \\ g_{k-1}^-(b; a) & \text{if } a > b \end{cases}$$

Proof. The proof is done in a similar way to the proof of Lemma 1. It is obvious that $g_k(l; a, b) = 0$ if either $l \ge \min\{a, b\}$, $a = b$, or (p_b, p_l, p_a) makes a right turn, as shown in the proof of Lemma 1. We thus assume that $l < \min\{a, b\}$, $a \neq b$ and (p_b, p_l, p_a) makes a right turn. In addition, if $k = 3$, then it is trivial that $g_3(l; a, b) = 1$ since there is a unique triangle counted by $g_3(l; a, b)$.

Now, suppose $k \ge 4$, let $\mathcal{G}$ be the set of all LT-monotone k-gons P such that $l(P) = p_l$ and the two corners adjacent to p_l are p_a and p_b on the lower and the upper chains of P, respectively. Note that $|\mathcal{G}| = g_k(l; a, b)$. We handled two distinct cases: $a < b$ or $a > b$.

First, assume that the former case where $a < b$. Let $\mathcal{G}'$ be the set of all LT-monotone $(k - 1)$-gons P' such that $l(P') = p_a$ and p_b is the corner adjacent to p_a on the upper chain of P'. Note that $|\mathcal{G}'| = g_{k-1}^+(a; b)$ by definition. In the following, we establish a one-to-one correspondence between $\mathcal{G}$ and $\mathcal{G}'$, concluding that $g_k(l; a, b) = |\mathcal{G}| = |\mathcal{G}'| = g_{k-1}^+(a; b)$, as claimed.

Consider any $P \in \mathcal{G}$. Since P is an LP-monotone k-gon, its leftmost triangle $\Delta p_l p_a p_b$ is completely contained in P, so we can cut it out from P to obtain an LP-monotone $(k - 1)$-gon P'. Observe that $l(P') = p_a$ and p_b is the corner adjacent to p_a on the upper chain of P'. This implies that $P' \in \mathcal{G}'$. So, we have $g_k(l; a, b) \le g_{k-1}^+(a; b)$.

Next, consider any $Q' \in \mathcal{G}'$, and let $Q := Q' \cup \Delta p_l p_a p_b$. Since $l(Q') = p_a$ and $\overline{p_a p_b}$ is a side of Q', Q indeed forms an LT-monotone k-gon. In addition, since $l < a < b$, observe that $l(Q) = p_l$ and the two corners adjacent to p_l are p_a and p_b on the lower and the upper chains of Q, respectively. Thus, Q is a member of $\mathcal{G}$, and therefore $g_k(l; a, b) \ge g_{k-1}^+(a; b)$. This completes the proof for the case of $a < b$.

The other case where $a > b$ can also be handled in analogously. □

As it was done above, we described a dynamic programming algorithm that computes $g_k = g_k(S)$ based on the recurrence shown in Lemma 2. In this case, we introduced another parameter m running from 3 to k, computed the values of $g_m^+(l; i)$ and $g_m^-(l; i)$ for all $1 \le l, i \le n$, and stored them in two-dimensional arrays G_m^+ and G_m^-, respectively.

Our algorithm is described as follows:

(1) Initialize G_m^+ and G_m^- with all the entries set to be zero for all $3 \le m \le k$.
(2) For each (l, a, b) with $1 \le l, a, b \le n$, if $l < \min\{a, b\}$, $a \neq b$, and (p_b, p_l, p_a) makes a left turn, then set $G_3^+[l, b] \leftarrow G_3^+[l, b] + 1$ and $G_3^-[l, a] \leftarrow G_3^-[l, a] + 1$
(3) For $m = 4, \ldots, k$ in this order, do the following:
 (3–1) For each (l, a, b) such that $1 \le l < a \neq b \le n$ and (p_b, p_l, p_a) make a left turn, let $t \leftarrow G_{m-1}^+[a, b]$ if $a < b$, or $t \leftarrow G_{m-1}^-[b, a]$ if $a > b$, and set $G_m^+[l, b] \leftarrow G_m^+[l, b] + t$ and $G_m^-[l, a] \leftarrow G_m^-[l, a] + t$.
(4) Compute the sum of all entries in G_k^+.
(5) Return it as the number $g_k(S)$ of all LT-monotone k-gons in S.

Theorem 2. *Given a set S of n points in the plane in general position and an integer $k \ge 3$, the number $g_k(S)$ of all LT-monotone k-gons in S can be computed in $O(kn^3)$ time and $O(n^2)$ space.*

Proof. We first proved the correctness of our algorithm. It is obvious that, after executing step (2) of our algorithm, it holds that $G_3^+[l; i] = g_3^+(l; i)$ and $G_3^-[l; i] = g_3^-(l; i)$ by Lemma 2.

Now, assume that we are currently in the main loop in step (3) of our algorithm for some m with $4 \le m \le k$, and suppose that $G_{m-1}^+[l, i] = g_{m-1}^+(l; i)$ and $G_{m-1}^-[l; i] = g_{m-1}^-(l; i)$ for all $1 \le l, i \le n$. For each (l, a, b) iterated in step (3–1), if $a < b$, then $g_m(l; a, b) = g_{m-1}^+(a; b)$; if $a > b$, then $g_m(l; a, b) = g_{m-1}^-(b; a)$ by Lemma 2. Thus, the variable t in step (3–1) stores the value of $g_m(l; a, b)$. Since $g_m^+(l; b) = \sum_a g_m(l; a, b)$ and $g_m^-(l; a) = \sum_b g_m(l; a, b)$, this implies that $G_m^+[l; i]$ and $G_m^-[l; i]$ store the correct values of $g_m^+(l; i)$ and $g_m^-(l; i)$, respectively, for all $1 \le l, i \le n$. This proves the correctness of our algorithm.

It is not difficult to see the time and space complexity of our algorithm described above is $O(kn^3)$ and $O(kn^2)$, respectively. The space complexity $O(kn^2)$ of our algorithm can be easily reduced to $O(n^2)$, because only the values of G_m^+ and G_m^- are necessary for the evaluation of $g_m(l; a, b)$. Hence, the theorem follows. □

Notably, our algorithm computes not only $g_k(S)$ but can also output the number $g_m(S)$ of LT-monotone m-gons for all $3 \le m \le k$, in the same time and space bound.

Corollary 1. *Given a set S of n points in the plane in general position and an integer $k \ge 3$, the number $g_m(S)$ of LT-monotone m-gons in S for every $3 \le m \le k$ can be computed in $O(kn^3)$ time and $O(n^2)$ space.*

IV. COUNTING LT-MONOTONE HOLES

For any $k \ge 3$, let $h_k(S)$ be the number of LT-monotone k-holes in S, and $h(S)$ be the number of all LT-monotone holes, that is, empty polygons in S. Note that $h(S) = h_3(S) + h_4(S) + \cdots + h_n(S)$.

As done in the previous section for the number $g_k(S)$ of LT-monotone k-gons in S, we start by defining the following quantities: for each $k \ge 3$ and each $1 \le l, a, b \le n$,

- let $h_k(l; a, b)$ be the number of all LT-monotone k-holes P in S such that the leftmost triangle of P is $\Delta p_l p_a p_b$ while p_a lies on the lower chain and p_b lies on the upper chain of P.
- let $h_k^-(l; a) := \sum_b h_k(l; a, b)$ be the number of all LT-monotone k-holes P in S such that $l(P) = p_l$ and p_a is the corner adjacent to p_l on the lower chain of P.
- let $h_k^+(l; b) := \sum_a h_k(l; a, b)$ be the number of all LT-monotone k-holes P in S such that $l(P) = p_l$ and p_b is the corner adjacent to p_l on the upper chain of P.

Also, we let

$$h(l; a, b) := \sum_{3 \le k \le n} h_k(l; a, b),$$

$$h^-(l; a) := \sum_{3 \le k \le n} h_k^-(l; a),$$

and

$$h^+(l; b) := \sum_{3 \le k \le n} h_k^+(l; b).$$

We then have the following recurrences for $h_k(l; a, b)$ and $h(l; a, b)$ for $1 \le l, a, b \le n$.

Lemma 3. *For any $k \ge 3$ and $1 \le l, a, b \le n$, it holds that*

$$h_k(l; a, b) = \begin{cases} 0 & \text{if } l \ge \min\{a, b\} \text{ or } a = b \\ 0 & \text{if } (p_b, p_l, p_a) \text{ makes a right turn} \\ 0 & \text{if } \Delta p_l p_a p_b \text{ is not empty} \\ 1 & \text{if } k = 3 \\ h_{k-1}^+(a; b) & \text{if } a < b \\ h_{k-1}^-(b; a) & \text{if } a > b \end{cases}$$

Proof. This recurrence can be verified by applying an almost identical argument used in the proof of Lemma 2, except that one should check that $h_k(l; a, b) = 0$ if the triangle $\Delta p_l p_a p_b$ contains a point of P in its interior, which is trivially true. □

Lemma 4. *For any $1 \le l, a, b \le n$, it holds that*

$$h(l; a, b) = \begin{cases} 0 & \text{if } l \ge \min\{a, b\} \text{ or } a = b \\ 0 & \text{if } (p_b, p_l, p_a) \text{ makes a right turn} \\ 0 & \text{if } \Delta p_l p_a p_b \text{ is not empty} \\ 1 + h^+(a; b) & \text{if } a < b \\ 1 + h^-(b; a) & \text{if } a > b \end{cases}$$

Proof. This recurrence can also be verified by applying an almost identical argument used in the proof of Lemma

1, or by using the fact that $h(l; a, b) = \sum_{3 \le k \le n} h_k(l; a, b)$ as follows. Unless $h(l; a, b) = 0$, it holds that

$$\begin{aligned} h(l; a, b) &= \sum_{3 \le k \le n} h_k(l; a, b) \\ &= 1 + h_3^+(a; b) + h_4^+(a; b) + \cdots + h_{n-1}^+(a; b) \\ &= 1 + h^+(a; b), \end{aligned}$$

if $a < b$ by Lemma 3; or, otherwise,

$$\begin{aligned} h(l; a, b) &= \sum_{3 \le k \le n} h_k(l; a, b) \\ &= 1 + h_3^-(b; a) + h_4^-(b; a) + \cdots + h_{n-1}^-(b; a) \\ &= 1 + h^-(b; a), \end{aligned}$$

if $a > b$ by Lemma 3. □

Based on the recurrences shown in Lemmas 3 and 4, one can devise dynamic programming algorithms, as it was done in the previous section, that run in $O(n^3)$ time to compute the number $h(S)$ of all LT-monotone holes in S and in $O(kn^3)$ time to compute the number $h_k(S)$ of LT-monotone k-holes in S for a given integer $k \ge 3$.

In the following, we show these algorithms can be improved in terms of the number $h_3(S)$ of empty triangles in S.

Let T be the set of triples (l, a, b) such that the triangle $\Delta p_l p_a p_b$ is empty and (p_b, p_l, p_a) makes a left turn. Note that $|T| = h_3(S)$. For each $1 \le l \le n$, let T_l be the set of pairs (a, b) such that $(l, a, b) \in T$. Dobkin et al. [5] presented an algorithm that enumerates all empty triangles in S in total $O(h_3(S))$ time.

Lemma 5 (Dobkin et al. [5]). *Given a set S of n points in general position, in $O(h_3(S))$ time, one can explicitly compute the set T and also the sets T_l for all $1 \le l \le n$.*

Our algorithm is fast; it computes the Number of all LT-monotone holes in S starts with applying Lemma 5 to specify T_l for all $1 \le l \le n$. Then, do the same as in the algorithm described in Theorem 1, except that the loop for each (a, b) in step (2–1) is repeated over T_l. This algorithm correctly computes $h(l; a, b)$, $h^-(l; a)$, and $h^+(l; b)$ by Lemma 4. Finally, note that the total number of iterations in step (2–1) is now bounded by $|T| = h_3(S)$. Hence, we conclude the following.

Theorem 3. *Given a set S of n points in the plane in general position, the number $h(S)$ of all LT-monotone holes in S can be computed in $O(h_3(S))$ time and $O(n^2)$ space.*

Similarly, we can modify the algorithm described in Theorem 2 to compute the number $h_k(S)$ of LT-monotone k-holes in S. In this case, we also start by applying Lemma 5 to specify the set T of triples (l, a, b). Then, do the same in the algorithm of Theorem 2, except that the loops for each (l, a, b) in steps (2) and (3–1) is now repeated over T. As above, we have the following results.

Theorem 4. *Given a set S of n points in the plane in general position and an integer $k \ge 3$, the number $h_k(S)$ of all LT-monotone k-holes in S can be computed in $O(k \cdot h(S))$ time and $O(n^2)$ space.*

Again, this algorithm computes the number $h_m(S)$ of LT-monotone m-holes for all $3 \le m \le k$.

Corollary 2. *Given a set S of n points in the plane in general position and an integer $k \ge 3$, the number $h_m(S)$ of LT-monotone m-holes in S for every $3 \le m \le k$ can be computed in $O(k \cdot h_3(S))$ time and $O(n^2)$ space.*

V. CONCLUDING REMARKS

For the first time, we have presented algorithms that count the exact number of LT-monotone polygons or holes in a given set S of n points. Our algorithms run in a dynamic programming fashion based on recursive structures on LT-monotone polygons, which were observed and proved in this paper.

There are two further directions for following research. First, can one efficiently count the number of monotone polygons and holes? The class of LT-monotone polygons is an proper subclass of all monotone polygons. So far, there is no known polynomial-time algorithm that counts the total number of monotone polygons or holes in a given set of points.

Second, what is the true computational complexity of this type of counting problems? The running times of our algorithms are roughly bounded by $O(kn^3)$ in the worst case. The author has proved a lower bound of $\Omega(n\log n)$ in [8] for the case of $k = 3$, that is, the problem of counting empty triangles. Can one improve this time bound into a subcubic time in the worst case? Or, can one prove a better lower bound than $\Omega(n\log n)$?

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Conflict of Interest(COI)

The authors have declared that no competing interests exist.

REFERENCES

1. P. Erdos and G. Szekeres, "A combinatorial problem in geometry," *Compositio Mathematica*, vol. 2, pp. 463-470, 1935.
2. G. Rote, G. Woeginger, B. Zhu, and Z. Wang, "Counting k-subsets and convex k-gons in the plane," *Information Processing Letters*, vol. 38, no. 3, pp. 149-151, 1991. https://doi.org/10.1016/0020-0190(91)90237-C
3. G. Rote and G. Woeginger, "Counting convex k-gons in planar point sets," *Information Processing Letters*, vol. 41, no. 4, pp. 191-194, 1992. https://doi.org/10.1016/0020-0190(92)90178-X
4. J. S. Mitchell, G. Rote, G. Sundaram, and G. Woeginger, "Counting convex polygons in planar point sets," *Information Processing Letters*, vol. 56, no. 1, pp. 45-49, 1995. https://doi.org/10.1016/0020-0190(95)00130-5
5. D. P. Dobkin, H. Edelsbrunner, and M. H. Overmars, "Searching for empty convex polygons," *Algorithmica*, vol. 5, pp. 561-571, 1990. https://doi.org/10.1007/BF01840404
6. I. Barany and Z. Furedi, "Empty simplices in Euclidean space," *Canadian Mathematical Bulletin*, vol. 30, no. 4, pp. 436-445, 1987. https://doi.org/10.4153/CMB-1987-064-1
7. M. Balko, M. Scheucher, and P. Valtr, "Holes and islands in random point sets," in *Proceedings of the 36th International Symposium on Computational Geometry (SoCG),* Zurich, Switzerland, 2020, pp. 14:1-14:16.
8. S. W. Bae, "Faster counting empty convex polygons in a planar point set," *Information Processing Letters*, vol. 175, article no. 106221, 2022. https://doi.org/10.1016/j.ipl.2021.106221
9. Y. H. Sung and S. W. Bae, "Counting convex and non-convex 4-holes in a point set," *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, vol. 104, no. 9, pp. 1094-1100, 2021. https://doi.org/10.1587/transfun.2020DMP0002
10. Y. H. Sung and S. W. Bae, "Counting non-convex 5-holes in a planar point set," *Symmetry*, vol. 14, no. 1, article no. 78, 2022. https://doi.org/10.3390/sym14010078
11. P. Erdos and G. Szekeres, "On some extremum problems in elementary geometry," *Annales Universitatis Scientiarium Budapestinensis de Rolando Eötvös Nominatae Sectio Mathematica*, vol. 3, no. 4, pp. 53-62, 1961.
12. G. Szekeres and L. Peters, "Computer solution to the 17-point Erdős-Szekeres problem," *The ANZIAM Journal*, vol. 48, no. 2, pp. 151-164, 2006. https://doi.org/10.1017/S144618110000300X
13. A. Suk, "On the Erdős-Szekeres convex polygon problem," *Journal of the American Mathematical Society*, vol. 30, no. 4, pp. 1047-1053, 2017. https://doi.org/10.1090/jams/869
14. A. F. Holmsen, H. N. Mojarrad, J. Pach, and G. Tardos, "Two extensions of the Erdős-Szekeres problem," *Journal of the European Mathematical Society (EMS Publishing)*, vol. 22, no. 12, pp. 3981-3995, 2020. https://doi.org/10.4171/jems/1000
15. P. Erdos, "Some more problems on elementary geometry," *Gazette* (*Australian Mathematical Society)*, vol. 5, no. 2, pp. 52-54, 1978.
16. H. Harborth, "Konvexe Fünfecke in ebenen Punktmengen," *Elemente der Mathematik*, vol. 33, pp. 116-118, 1978.
17. C. M. Nicolas, "The empty hexagon theorem," *Discrete & Computational Geometry*, vol. 38, pp. 389-397, 2007. https://doi.org/10.1007/s00454-007-1343-6
18. T. Gerken, "Empty convex hexagons in planar point sets," *Discrete & Computational Geometry*, vol. 39, pp. 239-272, 2008. https://doi.org/10.1007/s00454-007-9018-x
19. J. D. Horton, "Sets with no empty convex 7-gons," *Canadian Mathematical Bulletin*, vol. 26, no. 4, pp. 482-484, 1983. https://doi.org/10.4153/CMB-1983-077-8
20. I. Barany, and P. Valtr, "Planar point sets with a small number of empty convex polygons," *Studia Scientiarum Mathematicarum Hungarica*, vol. 41, no. 2, pp. 243-269, 2004. https://doi.org/10.1556/sscmath.41.2004.2.4
21. P. Valtr, "On the minimum number of empty polygons in planar point sets," *Studia Scientiarium Mathematicarum Hungarica*, vol. 30, pp. 155-163, 1995.
22. A. Dumitrescu, "Planar sets with few empty convex polygons," *Studia Scientiarum Mathematicarum Hungarica*, vol. 36, no. 1-2, pp. 93-110, 2000. https://doi.org/10.1556/sscmath.36.2000.1-2.9
23. R. Pinchasi, R. Radoicic, and M. Sharir, "On empty convex polygons in a planar point set," *Journal of Combinatorial Theory Series A*, vol. 113, no. 3, pp. 385-419, 2006. https://doi.org/10.1016/j.jcta.2005.03.007
24. O. Aichholzer, M. Balko, T. Hackl, J. Kyncl, I. Parada, M. Scheucher, P. Valtr, and B. Vogtenhuber, "A superlinear lower bound on the number of 5-holes," *Journal of Combinatorial Theory Series A*, vol. 173, article no. 105236, 2020. https://doi.org/10.1016/j.jcta.2020.105236
25. O. Aichholzer, R. Fabila-Monroy, H. Gonzalez-aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, and B. Vogtenhuber, "4-holes in point sets," *Computational Geometry*, vol. 47, no. 6, pp. 644-650, 2014. https://doi.org/10.1016/j.comgeo.2013.12.004
26. O. Aichholzer, R. Fabila-Monroy, H. Gonzalez-Aguilar, T. Hackl, M. A. Heredia, C. Huemer, J. Urrutia, P. Valtr, and B. Vogtenhuber, "On k-gons and k-holes in point sets," *Computational Geometry*, vol. 48, no. 7, pp. 528-537, 2015. https://doi.org/10.1016/j.comgeo.2014.12.007
27. R. Fabila-Monroy, C. Huemer, and D. Mitsche, "Empty non-convex and convex four-gons in random point sets," *Studia Scientiarum Mathematicarum Hungarica*, vol. 52, no. 1, pp. 52-64, 2015. https://doi.org/10.1556/sscmath.52.2015.1.1301

Sang Won Bae https://orcid.org/0000-0002-8802-4247

Sang Won Bae received his Ph.D. in 2008 at Department of Computer Science, Korea Advanced Institute of Science and Technology, Daejeon, Korea. At present, he is working as a professor at Division of Artificial Intelligence and Computer Science, Kyonggi University, Suwon, Korea. Research interests include algorithms design and analysis in computational geometry, discrete and combinatorial geometry, graph theory, and their algorithmic applications to other disciplines.